MATHEMATICS

DISTANCE EDUCATION MODULE

GRADE 10

MODULE 1

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FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA
MINISTRY OF EDUCATION

MATHEMATICS

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Module 1

Relations and Functions, Polynomial Functions Exponential and Logarithmic Functions

Authors:

Tadele Mekonnen Yihun (MSc)

Mamo Teketel Mekasha (MSc)

Editors:

Akalu Chaka Mekuria (MA)   **(Curriculum Editor)**

Endalfer Melese Moges (MA)   **(Language Editor)**

Designer:

 Aknaw H/mariam Habte (MSc)

**FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA
MINISTRY OF EDUCATION**

HAWASSA UNIVERSITY

❈ Introduction

Dear distance learner, **Grade 10 mathematics** has two modules. The first module consists 3 units. They are: Relations and Functions, Polynomial functions, Exponential and Logarithmic Functions. The second module has 4 units: Trigonometric Functions, Circles, Solid Figures, Coordinate Geometry. Each unit is generally organized as: Activities, Definitions, Checklist, Self-Test Exercise and Unit Review Exercises.

Each unit starts with its own introduction and objectives. Each unit is further broken in to sections. In each section, sufficient number of examples are given to make the module simple and understandable.

Each section is followed by checklist to help you think about you have studied. This is an important technique to review the section before you start working on the self-test exercise.

At the end of each section, we have self-test exercise, your achievement in the exercise will show you how much of the given section you have mastered. Answer key for all activities and self-test exercises is given at the end of the module. Before looking at the answer key you have to try your best to find the solution individually or in your study group.

Generally, the two Grade 10 Mathematics Modules are expected to be covered in one academic year. Thus, you will have about six months to complete each module. As you are learning by your-self, you will not have regular direct face-to-face support from anyone. Therefore, you are advised to adjust your time to go through each unit again and again until you feel comfortable with it and do all the tasks and assignments in time by your-self. However, whenever you encounter any challenges, you may ask your tutor/tutorial center for assistance.

Dear distance learner! you will have at least one tutor marked assignment which accounts about 30% and one final examination valuing 70% which you will be delivered by the respective tutorial center.

◎ Learning Outcomes

By the end of this module, you will be able to:

- ✺ Define relations and functions.
- ✺ Identify types of functions.
- ✺ Sketch graphs of functions.
- ✺ Define polynomial function.
- ✺ Perform the four-fundamental operation on polynomials.
- ✺ Apply theorems on polynomial functions to solve related problems.
- ✺ State and apply the properties of exponents and logarithms.
- ✺ Define exponential and logarithmic functions

✳ Draw graphs of exponential function and logarithmic functions.

✳ Describe the properties of exponential and logarithmic functions by using their graphs

There are a number of symbols in this material to guide you as you study

This tells you there is objective(s) of the unit and subunit or competencies sections.

This tells you there is an overview of the unit and what the unit is about

This tells you there is an in-text question to answer or think about in the text

This tells you to take not of or to remember an important point

This tells you there is a self-test for you to do

This tells you there is a checklist

This tells you there is a written assignment

This tells you that this is the key to the answers for the self-tests

This tells you there is an activity for you to do

Contents

UNIT 1

RELATIONS AND FUNCTIONS

❈ Introduction

Dear learner, in order to continue our study of functions, we introduce the more general idea of a relation. As its name suggests, the concept of a relation is a familiar one. In our daily life, we come across many patterns that characterize relations with brothers and sisters, mother and daughters, father and sons, teachers and students etc. In mathematics also, we come across many relations such as number m is greater than number n; line n is perpendicular to line m etc. The concept of relation is established in mathematical form. The word "function" is introduced by Leibnitz in 17th century. *Function is defined as a special type of relation.* In the present unit, we shall discuss Cartesian coordinates, conditions for a relation to be a function, different types of functions and their properties.

Learning Outcomes

By the end of this section, you will be able to:
  * define relations.
  * define functions.
  * identify types of functions.
  * sketch graphs of various types of relations and functions.

Unit contents

Section 1: Definition of Relation

Section 2: Functions

Section 3: Types of Functions

Section 4: Application of Relations and Functions

Unit Learning Strategies

We would like to mention the following for your study of this chapter.

- Create a schedule and follow it. Scheduling makes you more practical.

- New topics are explained and worked out examples are given. So, take your time to read and do the examples and the activities given in the module by your own repeatedly until you get the steps clearly and apply it in self-test exercises.

- Try your best to understand the concept of functions, their properties and their graphical interpretations. Because, this can support your understanding in unit 2, 3 and 4.

- When you attain what you had intended for a specific learning session, you should reward yourself with anything positive. This makes you more inspired to keep moving and cherish your learning and achievements.

| Section 1 | Definition of relation |

❈ Section Overview

Dear learner, you are familiar with Cartesian product of two sets in Grade 9. The Cartesian product of any two sets A and B is a set, and the elements of that set are ordered pairs. In each ordered pair, the first component is an element of A, and the second component is an element of B. Relations and its types concepts are one of the important topics of set theory. Sets, relations and functions are interlinked topics. Sets denote the collection of ordered elements whereas relations and functions define the operations performed on sets.

The relations define the connection between the two given sets. There are also different types of relations stating the connections between the sets. Hence, here we will learn about relations that we are going to define now. In addition, terms like domain, range and graphs of a relation will be defined.

Section Objectives

By the end of this section, you will be able to: -

- define relation.

- draw the graphs of relations.

Do you remember the cartesian product or cross product of two sets? Can you write the first and the second element of these sets?

Activity 1.1

Let $A = \{2, 3, 5, 7, 8\}$ and $B = \{10, 7, 9, 6, 4\}$. List all the ordered pairs (x, y) which satisfy each of the following sentences where $x \in A$ and $\in B$.

    a. x is greater than y.   **c.** y is a multiple of x.

    b. The sum of x and y is odd.   **d.** x is half of y.

Have you done it? Well. If not, you can find the answer at the end of this module.

The **Cartesian product** of two sets A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) where a is in A and b is in B. In terms of set-builder notation, that is

$A \times B = \{(a, b): a \in A \text{ and } b \in B\}$. a is the first element and b is the second element.

As you have noticed from above activity, each sentence involves what is intuitively understood to be a relationship. Expressions of the type "is greater than", "is a multiple of", "is half of", etc. which express the relation are referred to as related phrases.

1.1 Basic concepts of relations

In order to continue our study of functions, we introduce the more general idea of a relation, as its name suggests, the concept of relation is a familiar one. Everyone has relatives or relations -father, mother, brothers, and sisters etc. What may come as a surprise, however, is that this concept has an important place in mathematics. Let us start with an example taken from everyday life in the relationship of fatherhood. The problem we set for ourselves is to describe this relation in mathematical terms.

If you think about this problem for a few minutes, you will probably find it somewhat more difficult than you expected. Although everyone knows what it means to say that y is a father of x, it is not quite so clear how to put this in the language we use in mathematics.

In our daily life, we usually talk about relations between various things. For example, 7 is less than 9; Addis Ababa is the capital city of Ethiopia; Walia Ibex is endemic to Ethiopia, and so on. In all these cases, we find that a relation involves pairs of objects in some specific order. In this unit, you will learn how to link pairs of objects from two sets and then introduce relations between the two objects in the pair. You also learn here about special relations which will qualify to be functions.

In activity 1.1 we have observed the following:

    1.   In the case of relations between objects, order is important.

    2.   A relation establishes pairing between objects.

Therefore, from a mathematical standpoint, the meaning of a relation is given below.

> ### Definition 1.1
>
> A relation is a set of ordered pairs. It is denoted by R.

Example 1

Given a relation $R = \{ (x, y): y \text{ is greater than } x\}$, which of the following ordered pairs belong to this relation R?

$(2, 4), (4, 3), (1.1, 1.11), (1.1, -3), (-5, -3), (7, 7), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{2}{3}, \frac{3}{2}\right), (0.45, 0.46)$

Solution:

$R = \{(2, 4), (1.1, 1.11), (-5, -3), \left(\frac{2}{3}, \frac{3}{2}\right), (0.45, 0.46)\}$.

Example 2

Let $R = \{(x, y): y \text{ is a multiple of } x\}$, then which of the following ordered pairs belong to R ?

$(2, 4), (4, 3), (3, 9), (18, -3), (9, 3), (7, 7), (3, 12), (6, 18), (30, 5)$

Solution:

$R = \{(2, 4), (3, 9), (7, 7), (3, 12), (6, 18)\}$.

Example 3

Let $R = \{(x, y): y = x^2\}$.

a. Find the ordered pairs belong to R which have the following first entries: $0, 1, -1, -2, \frac{1}{5}, 3, -3$.

b. Find the ordered pairs belong to R which have the following second entries: $4, 1, 0, \frac{1}{4}, \frac{1}{25}$.

Solution:

a. $R = \{(0, 0), (1, 1), (-1, 1), (-2, 4), \left(\frac{1}{5}, \frac{1}{25}\right), (3, 9), (-3, 9)\}$.

b. $R = \{(2, 4), (1, 1), (0, 0), \left(\frac{1}{2}, \frac{1}{4}\right), \left(\frac{1}{5}, \frac{1}{25}\right)\}$.

1.2 Domain and range of a relation

Any set of ordered pairs (x, y) is called a relation in x and y.

Activity 1.2

For the relation $R = \{(4, 2), (-3, 5), (6, 3), (-3, 4)\}$ determine the set of all the first elements of R and the set of all the second elements of R.

The first ordered pairs of R are 5, -2 and the second ordered pairs of R are 2, 3, 4.

> ☞ **Definition 1.2**
>
> Let R be a relation from a set A to a set B. Then
>
> i. Domain of $R = \{x: (x, y) \text{ belongs to R for some } y\}$
>
> ii. Range of $R = \{y: (x, y) \text{ belongs to R for some } x\}$

Example 1

Determine the domain and range of the relation $R = \{(-2, 1), (-1, \ 0), (0, 0), (4, 2), (3, 5)\}$.

Solution:

The domain of the relation $R = \{-2, -1, 0, 4, 3\}$. The range of the relation $R = \{1, 0, 2, 5\}$.

Example 2

Write some ordered pairs that belong to the relation $R = \{(x, y): y = 2x; \ x, \ y \in \mathbb{Z}\}$. Find also the domain and range of R.

Solution:

$R = \{(-2, -4), (-1, \ -2), (0, 0), (1, 2), (2, 4)\}$.

The domain of the relation R is the set of all integers and the range of the relation R is the set of all even integers.

Example 3

Find the domain and the range of each of the following relations:

   a. $R = \{(x, y): \ y \text{ is the square } x\}$.

   b. $R = \{(x, y): \ y \text{ is the square root of } x\}$.

Solution:

   a. Domain of relation $R = \{x: x \in \mathfrak{R}\}$ and Range of relation $R = \{y: y \geq 0\}$.

   b. Domain of relation $R = \{x: x \geq 0\}$ and Range of relation $R = \{y: y \geq 0\}$.

1.3 Graphs of Relation

Dear learner, you have understood what a relation is and how it can be described using Cartesian products. You will now see how relations can be represented through graphs. You may graphically represent a relation R from domain to range by locating the ordered pairs in a coordinate system.

> **?**
> i.   What is a coordinate system?
> ii.   What is a Cartesian coordinate system (or xy-coordinate system)?
> iii.   What is the coordinate of the origin?

Co-ordinate system is used to locate the position of a point in a plane using two perpendicular lines. Points are represented in the form of coordinates (x, y) in two-dimension with respect to x- and y-axes. The Cartesian coordinate system in two dimensions (also called a rectangular coordinate system) is defined by an ordered pair of perpendicular lines (axes). It has a single unit of length for both axes and an orientation for each axis. The point where the axes meet is taken as the origin for both and used as a turning point for each axis into a number line. For any point P, a line drawn through P perpendicular to each axis and the position where it meets the axis is interpreted as a number. The two numbers in that chosen order are the cartesian coordinate of P. The coordinates are usually written as two numbers in parentheses in that order separated by a comma, as in $(3, 2)$. Thus the origin has coordinates $(0, 0)$, and the points on the positive half-axes which is one unit away from the origin and have coordinates $(1, 0)$ and $(0, 1)$.

From section 1.2, remember that the set of ordered pairs (x, y) of real numbers such that x is in the domain of the relation and y is in the range of the relation is represented by the set of points in the xy-coordinate.

Since an ordered pair of real numbers can be pictured as a point on the graph, we can use graphs to represent relations R as:

$R = \{(-3, -3), (-3, 3), (-2, 2), (-1, 2), (1, 2), (3, 2), (-2, -2), (2.6, 0.6), (1, 0.6)\}$
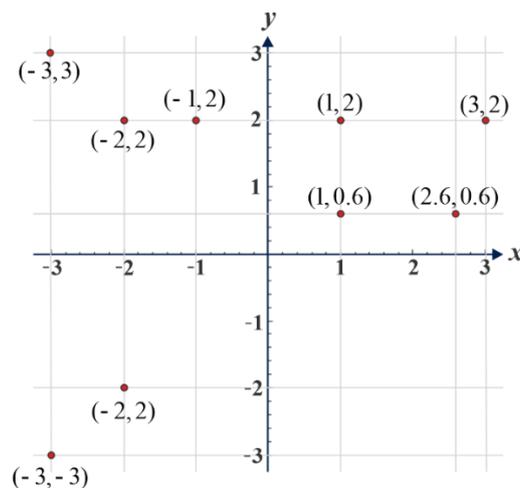
Figure 1.1

Example 1

Sketch the graph of the relation R if $R = \{(x, y): y = x\}$.

Solution:

We take the values of x, calculate the corresponding values of y, plot the resulting points (x, y) and connect the points.

Table 1.1

x	-2	-1	0	1	2
y	-2	-1	0	1	2
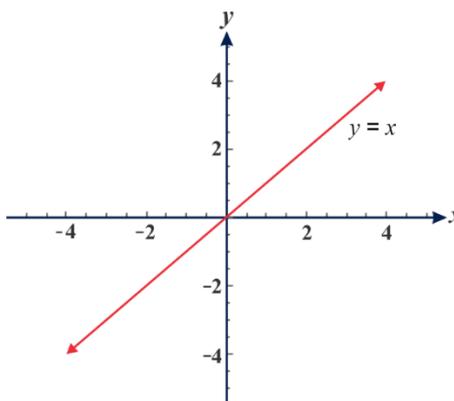
Figure 1.2 the graph of $y = x$

In general, to sketch graphs of relations involving inequalities, do the following steps

1. Draw the graph of the line(s) in the relation on the xy-coordinate system.
2. If the relating inequality is $\leq\ or\ \geq$, use a solid line; if it is $<\ or\ >$, use a broken line.
3. Then take arbitrary ordered pairs represented by the points.
4. The region that contains these points representing the ordered pair satisfying the relation will be the graph of the relation.

Example 2

Sketch the graph of the relation R if $R = \{(x, y): y > x\}$.

Solution:

To sketch the graph

1. Draw the line $y = x$.
2. Since the relation involves $y > x$, use the broken line: "the points on the line $y = x$ are not included."
3. Take points representing ordered pairs, say $(0, 3)$ and $(0, -2)$ from above and below the line $y = x$.
4. The ordered pair $(0, 3)$ satisfies the relation. Hence, points above the line $y = x$ are members of the relation R.
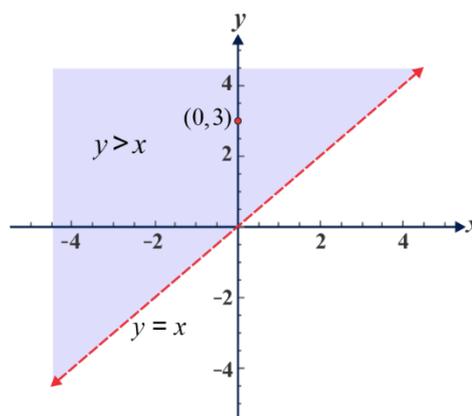
Figure 1.3 the graph of the relation $y > x$.

Example 3

Sketch the graph of the relation R if $R = \{(x, y): y \leq x + 2\}$.

Solution:

1. Draw the line $y = x + 2$.

2. Since the relating inequality is $\leq$ use the solid line.

3. Select two points representing ordered pairs one from one side of line $y = x + 2$ and another from the other side of the line. For example, points with coordinates $(0, 4)$ and $(1, 0)$. Obviously, $(1,0)$ satisfies the relation R is $y \leq x + 2$, as $0 \leq 1 + 2$.

4. Shade the region below the line, $y = x + 2$ which contains the point $(1, 0)$. Hence, the graph is the shaded region in figure1.8.
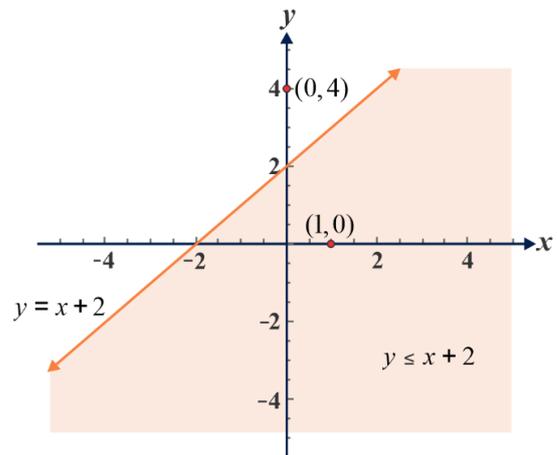
Figure 1.4 the graph of the relation $y \leq x + 2$.

Example 4

Sketch the graph of the relation R and determine its domain if

$R = \{(x, y) : y < x \text{ and } y > -x + 2\}$.

Solution:

Sketch the graphs of $y < x$ and $y > -x + 2$ on the same coordinate system. The two regions have some overlap. The intersection of the two regions is the graph of the relation. So, taking only the common region, we obtain the graph of the relation as shown in figure 1.5.

To obtain the intersection point solve the equations $y = x$ and $y = -x + 2$.

Note that these two lines divide the coordinate system into four regions. Take any points one from each region and check if they satisfy the relation. Say $(2, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$.

The point $(2, 0)$ satisfies both inequalities of the relation. So, the graph of the relation is the region that contains $(2, 0)$. Hence, Domain of the relation is the set of real number x: and $x > 1$ and Range of R is the set of all real numbers y.
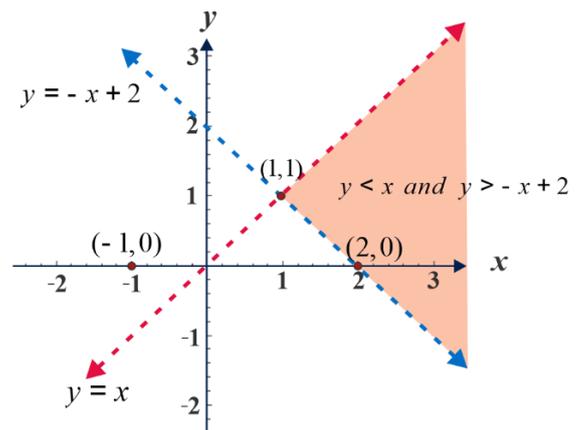
Figure 1.5 the graph of the relation R

Example 5

From the graph of each of the following relations, represented by the shaded region, specify the relation and determine the domain and the range:
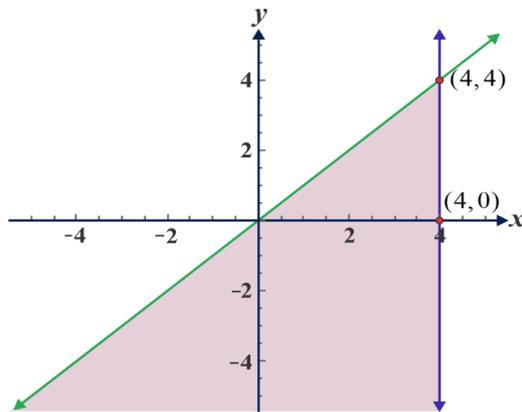
Figure 1.6

Solution:

$$R = \{(x, y): y \leq x \text{ and } x \leq 4\}; \text{ Domain } = \{x: x \leq 4\} \text{ and Range } = \{y: y \leq 4\}.$$

Checklist

Dear learner, now it is time to check your understanding about relation. Read each question and put a tick (" $\checkmark$ ") if you can do the task and an "x" if you can't on the space provided.

- Can you define a Cartesian product?   □
- Can you define a relation?   □
- Can you define a coordinate system?   □
- Can you find the domain of the relations?   □
- Can you find the range of the relations?   □
- Can you give two examples of relations?   □
- Can you define a Cartesian coordinate system?   □
- Can you draw graphs of relations?   □

If your response to the above check list questions is positive, proceed to do the self-test exercise given below and check your answer at the end of your module. If not please review the section again and again till you think you do not have any problem.

Self-Test Exercise 1

1. Given a relation $R = \{(x, y): y \text{ is greater than } x\}$.

   a. Which of the following ordered pairs belong to the relation?

   $$(2, 1), (-4, 3), (-2, 0), (0.2, 0.21), (-0.2, -0.21), (7, 7), (-2, -3), (0, 5)$$

2. If $R = \{(x, y): y = 2x - 5\}$, then list some of the ordered pairs belong to R.

3. Let $R = \{(x, y): y = -x^3\}$.

a. Find the ordered pairs belong to R which have the following first entries: $0, 1, 2, -2, \frac{1}{5}, 3, -3$.

b. Find the ordered pairs belong to R which have the following second entries: $8, -1, -8, -27, \frac{1}{27}$.

4. Determine the domain and range of the relation

$$R = \{(-2, \ 5), (1, \ 6), (3, \ 4), (4, \ 4), (5, -3)\}.$$

5. Write some ordered pairs that belongs to the following relation R; and find also the domain and range of this relation.

a. $R = \{(x, y) : y = -3x; \ x, y \in \mathbb{Z}\}.$ **b.** $R = \{(x, y) : y = 2x; \ x, y \in \mathbb{Z}\}.$

6. For each of the following relations, sketch the graph.

a. $R = \{(x, y) : y > x + 2\}.$ **b.** $R = \{(x, y) : y \geq -x + 1\}.$

c. $R = \{(x, y) : y > 2x - 3\}.$ **d.** $R = \{(x, y) : y < 3x - 2\}.$

Section 2 | Function

❈ Section Overview

Dear learner, in this section, you shall learn about special types of relations which are called functions, the domain and range of function, the graphs of functions and combination of functions. The concept of function is the most important point in mathematics. There are terms such as 'map' or 'mapping' used to denote a function.

Section Objectives

By the end of this section, you will be able to: -

- define a function.
- find domain of a function.
- find range of a function.
- find sum, difference, product and division of functions.
- draw the graphs of different functions.

2.1 The notion of a function

Dear learner, to introduce the notion of function, we begin by defining a relation between a set of inputs having one output each. We usually visualize one of them is an input and the other as an output. The input is called the first ordered pair, the second ordered pair is the output.

? How do you define an ordered pair?

An ordered pair is a combination of the x coordinate and the y coordinate, having two values written in a fixed order within parentheses.

The ordered pair (2, 5) means a pair of two integers, strictly in the order with 2 at the first place and 5 at the second place.

> **?** How do you obtain the ordered pair that corresponds to a given set?

The pair of elements that occurs in a particular order and is enclosed in brackets is known as a set of ordered pairs. If 'a' and 'b' are two elements, then the two different pairs are (a, b), (b, a). In an ordered pair (a, b), a is called the first element, and b is called the second element.

> **?** What are relations?

A relation defines the relationship between sets of values of ordered pairs.

For example, $y = x + 3$ and $y = x - 1$ are relations because every x-value produces a different y-value. A relation is any set of ordered-pair numbers. Consider the following arrow diagram.
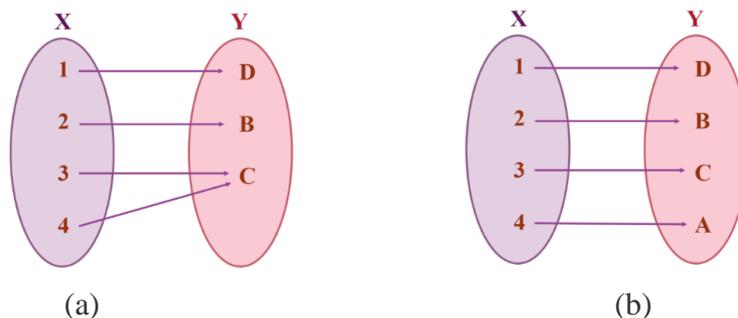
Figure 1.7

This mapping depicts a relation from set X into set Y. A relation from X to Y is a subset of $X \times Y$. In figure (a), the ordered pairs are $(1, D), (2, B), (3, C), (4, C)$. In figure (b), the ordered pairs are $(1, D), (2, B), (3, C), (4, A)$.

Activity 1.3

Consider the following relations:

$R_1 = \{(2, 3), (4, 5), (3, 6), (6, 7), (5, 8)\}$,

$R_2 = \{(2, 3), (4, 3), (3, 6), (7, 6), (5, 8)\}$

$R_3 = \{(2, 3), (2, 5), (3, 6), (3, 7), (5, 8)\}$

Construct the arrow diagram.

a. How are the first elements of the ordered pair related with the second elements of the ordered pair?

b. In each relation, are there ordered pairs with the same first coordinate?

Have you done it? Well.

? What is a function?

A function is a relationship between inputs where each input is related to exactly one output. In activity1.1, does each input is related to exactly one output for R_1? For R_2? For R_3?

In R_1 and R_2 each input is related to exactly one output. But R_3 is not since one input maps to two outputs. i.e., 2 maps with 3 and 5.

Definition 1.3

A function f is a set of ordered pairs with the property that whenever (x, y) and (x, z) belong to f, then $y = z$.

Example 1

Consider the following relation R,

$R = \{(4, 5), (6, 6), (3, 1), (9, 7), (5, 2)\}$. This relation is a function because no two distinct ordered pairs have the same first element.

Example 2

Consider the relation R.

$R = \{(2, 3), (5, 3), (3, 6), (7, 6), (5, 8)\}$. Since $(5, 3)$ and $(5, 8)$ belong to the relation R and $3 \neq 8$ the relation R is not a function.

Example 4

Consider the following arrow diagrams in figure1.8
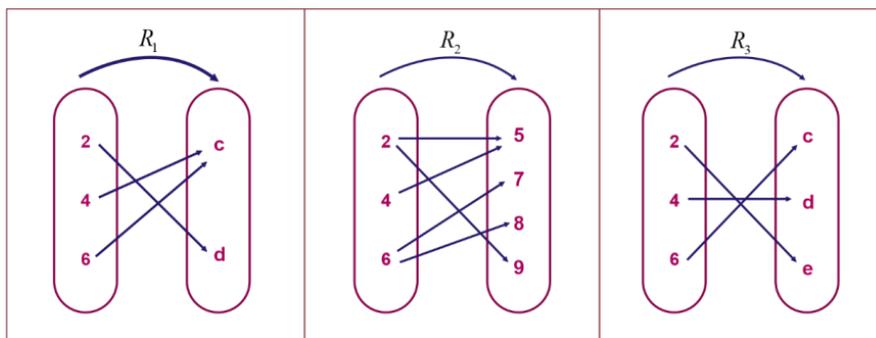
Figure 1.8

Which of the relations R_1, R_2 and R_3 represent y as a function of x?

Solution:

R_1 and R_3 are functions, but R_2 is not a function because 2 and 6 are both mapped onto two numbers.

Note

f, g and h are the most commonly used letters to represent a function; however, any letter of the alphabet can be used.

2.2 Domain, Codomain and Range of a Function

Activity 1.4

Consider the following arrow diagram of a function f and find the algebraic rule for f.
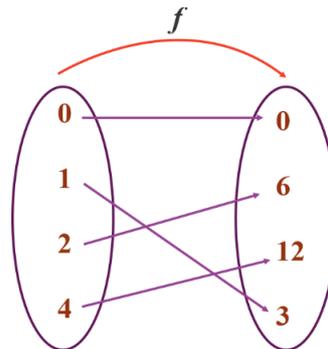
Figure 1.9

The domain of a function f is the set of all values of x for which f is defined and this corresponds to all of the x-values on the graph in the xy -plane. Domain – The set of all possible values which qualify as inputs to a function is known as the domain of the function. In other words, the domain of a function can be defined as the entire set of values possible for independent variables.

Co-Domain – The set of all the outputs of a function is known as the range of the function or after substituting the domain, the entire set of all values possible as outcomes of the dependent variable. The range of the function f is the set of all values $f(x)$ which corresponds to the y values on the graph in the xy-plane.

Notation: If x is an element in the domain of a function f, then the element in the range that is associated with x is denoted by $f(x)$. This is called an image of x under the function f. The notation $f(x)$ is referred to function value and we read as ' f of x' and x is the pre-image of $f(x)$. We can define a function $f(x) = 2x$ with a domain and codomain of integers. But by thinking about it we can see that the range is just the even integers. The range is a sub-set of the co-domain

Example 1

For each of the following functions, find the domain, co-domain and the range.

a. $F = \{(3, -2), (5, 4), (1, 2), (-3, 7)\}$

b. $R = \{x \in \mathcal{R} : y = x\}$.

Solution:

    a. Domain of $F = \{-3, 1, 3, 5\}$, Co-domain of $F = \{-2, \; 2, \; 4, 7\}$ and the range of

        $F = \{-2, \; 2, \; 4, 7\}$.

    b. Domain of R is a set containing all real numbers x and the co-domain and range is the set

        containing all real numbers y.

A function from A to B can sometimes be denoted as $f : A \rightarrow B$, where the domain of the function f

is A and the range of the function f is contained in B, in which we say B contains the image of the

elements of A under the function of f.

Example 2

Consider $f(x) = 3x + 4$. Find the domain and the range of the function f.

Solution:

Since $f(x) = 3x + 4$ is defined for every real number x, the domain of the function is the set of all

real numbers. The range is also the set of real numbers since every real number y has a pre-image of

a real number x such that $y = f(x) = 3x + 4$.

Example 3

If $f(x) = \sqrt{x - 2}$, then find the domain and range of f.

Solution:

Since the expression in the radical must be non-zero, $x - 2 \geq 0$. This implies $x \geq 2$. Hence,

the domain of the function is the set of real numbers greater than or equal to 2. The range of

the function is greater than or equal to zero.

> If $f : A \rightarrow B$ is a function, then, for any x included in A (the first coordinate), the image of x under f, $f(x)$ is called the functional value of f at x.

For example, if $f(x) = x + 5$, then f at $x = 4$ is $f(4) = 4 + 5 = 9$.

Finding the functional value of f at x is called evaluating the function at x.

Example 4

For the function $f(x) = 1 - 3x$,

    a. determine the domain and range of f.

    b. find the value of $f(2)$ and $f(-1)$.

Solution:

    a. The function $f(x) = 1 - 3x$ is defined for every real number x ; the domain and   range of

        the function f is the set all of real numbers.

b. $f(2) = 1 - 3(2) = 1 - 6 = -5$ and $f(-1) = 1 - 3(-1) = 1 + 3 = 4.$

2.3 Combination of functions

Dear learner, in this section you will see, just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two functions can also be combined to create a new function.

Activity 1.5

Consider the functions $f(x) = 2x + 3$ and $g(x) = -2$. Then,

    a. find $f + g$; $f - g$; $f \cdot g$ and $\frac{f}{g}$.

    b. determines the domain and range of f and g

A. Sum of functions

> Let f and g be two functions with overlapping domains. Then, for all x common to both domains, the sum of f and g defined as follows:
> $$(f + g)(x) = f(x) + g(x)$$

Example 1

Given $f(x) = 2x + 4$ and $g(x) = 3x - 1$. Find $(f + g)(x)$ and evaluate the sum when $x = 2$.

Solution:

We know from above the sum of functions f and g is

$$(f + g)(x) = f(x) + g(x)$$
$$= (2x + 4) + (3x - 1) = 5x + 3$$

And, $(f + g)(2) = 5(2) + 3$
$$= 10 + 3 = 13$$

B. Difference of functions

> Let f and g be two functions with overlapping domains. Then, for all x common to both domains the difference of f and g is defined as follows:
> $$(f - g)(x) = f(x) - g(x).$$

Example 2

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$. Find $(f - g)(x)$ and evaluate the difference when $x = 2$.

Solution:

The difference of the function f and g is

$$(f - g)(x) = f(x) - g(x)$$
$$= (2x + 1) - (x^2 + 2x - 1)$$
$$= -x^2 + 2.$$

$$\text{and, } (f - g)(2) = -2^2 + 2$$
$$= -4 + 2 = -2.$$

C. Product of functions

☞ Let f and g be two functions with overlapping domains. Then, for all x common to both domains the product of f and g is defined as follows:
$$(f \cdot g)(x) = f(x) \cdot g(x)$$

Example 3

Given: $f(x) = x^2$ and $g(x) = x - 3$, find $(f \cdot g)(x)$ and then evaluate the product when $x = 4$.

Solution:

$$(f \cdot g)(x) = f(x) \cdot g(x)$$
$$= x^2(x - 3)$$
$$= x^3 - 3x^2.$$

If $x = 4$, the value of this product is $(f \cdot g)(4) = 4^3 - 3(4^2) = 64 - 48 = 16$.

? Can you check $\quad x^2 (x - 3) = x^3 - 3x^2$?

D. Quotient of functions

☞ Let f and g be two functions with overlapping domains. Then, for all x common to both domains the quotient of f and g is defined as follows:
$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}, \; g(x) \neq 0.$$

Example 4

If $f(x) = 4$ and $g(x) = x - 3$, then the quotient of f and g is
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{4}{x-3}.$$

The domain of combination of functions

Activity 1.6

Let $f(x) = x + 2$ and $g(x) = x^2$, then find

    a. $(f \cdot g)(x)$ and $(f \cdot g)(-1)$ **b.** $\left(\frac{f}{g}\right)(x)$ and $\left(\frac{f}{g}\right)(3)$ **c.** $\left(\frac{g}{f}\right)(x)$ and $\left(\frac{g}{f}\right)(3)$

The domain of an arithmetic combination of functions f and g $(f + g, f - g)$ consists of real numbers common to the domain of f and g.

? What is the domain of multiplication and division of functions?

As with the addition and subtraction of functions, the domain for the multiplication and division of functions is the intersection $f \cap g$. However, when we divide functions, we need to further restrict the domain of the combined function since we can't divide by zero. So, for the division of functions, the domain is $\{x: x \epsilon (f \cap g) \ and \ g(x) \neq 0\}$ .This is read as "the set of all values of x such that x is an element of the intersection of f and , as long as g is not equal zero."

In the case of quotient of functions $\frac{f(x)}{g(x)}$, there is further restriction that $g(x)$ is not equal to zero.

Example 1

Given $f(x) = x - 5$ and $g(x) = x + 1$. Find the domain of $(f + g)\,(x)$ and $(f - g)(x)$.

Solution:

The domain f and g is the set of all real numbers. So, the domain of $f + g$ and $f - g$ is the set of all real numbers.

Example 2

Given: $f(x) = 2$ and $g(x) = x^2$, find $(f \cdot g)(x)$ and $\frac{f(x)}{g(x)}$ and their domains.

Solution:

The domain f and g is the set of all real numbers. So, the domain of $f \cdot g$ and $\frac{f}{g}$ is the set of all real numbers. Here, g cannot be zero. So, the domain of $\frac{f}{g}$ is $x \epsilon \mathfrak{R}: x \neq 0$ (the set of real numbers without zero) since g cannot be zero.

Example 3

Find $\frac{f}{g}(x)$ and $\frac{g}{f}(x)$ for functions given by $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x^2}$. Then, find the domain

of $\frac{f}{g}$ and $\frac{g}{f}$.

Solution:

The quotient of f and g is given by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{4-x^2}}$ and the quotient of g and f is

$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{4-x^2}}{\sqrt{x}}$.

The domain of f is $[0, \infty)$ and the domain of g is $[-2, 2]$. The intersection of these domains is

$[0, 2]$. Here, g cannot be zero. So, the domain of $\frac{f}{g}$ is $[0,2)$ and domain of $\frac{g}{f}$ is $(0,2]$ since f cannot

be zero.

☞

> A numerical relation R is a function if and only if no **vertical line** in the plane
>
> intersects the graph of R in more than one point.

Vertical Line Test:

Vertical line test is used to determine whether a graph of a curve is a function or not. If any curve

cuts a vertical line at more than one point then the curve is not a function.

Example 1

The figure1.10 describes the graph is not a function.
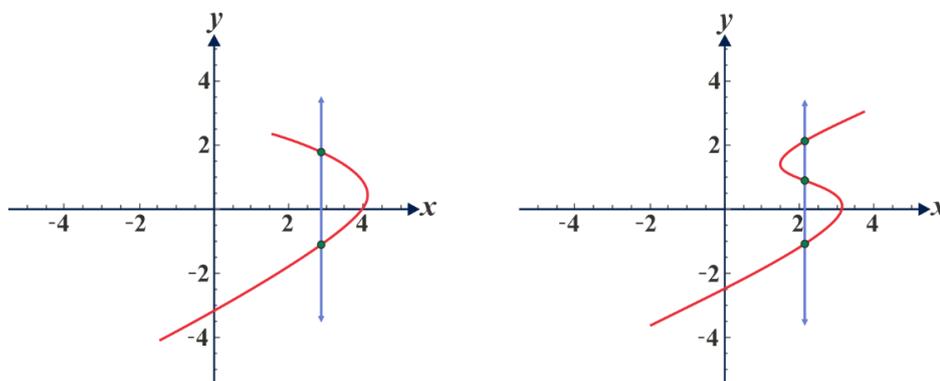
Figure 1.10

Checklist

Dear learner, now it is time to check your understanding about relation. Read each question and

put a tick (" ✓ ") if you can do the task and an "x" if you can't on the space provided.

- Can you define a function?   ☐

- Can you define a relation? ☐
- Can you draw the graphs of functions? ☐
- Can you find the domain of the functions? ☐
- Can you find the range of the functions? ☐
- Can you give two examples of functions? ☐
- Can you explain the rule for combination? ☐
- Can you find combination of functions? ☐
- Can you find the domain of combination of functions? ☐
- Can you find the range of combination of functions? ☐

If your response to the above check list questions is positive, proceed to do the self-test exercise given below and check your answer at the end of your module. If not please review the section again and again till you think you do not have any problem.

✗ Self-Test Exercise 2

1. Determine whether each of the following relation is a function or not, and give reasons for those that are not functions.
   a. $R = \{(-5, 4), (2, 8), (1, 6), (-1, 0), (-3, 4)\}$.
   b. $R = \{(3, -7), (4, 7), (1, -4), (2, 0), (4, 1)\}$.
   c. $R = \{(x, \ y): y \text{ is a multiple of } x\}$.
   d. $R = \{(x, \ y): y^2 = x\}$.

2. Find the domain and the range of F where $F = \{(-2, 1), (0, 0), (4, -2), (5, -3)\}$.

3. Given $f(x) = 3x + 2$ and $g(x) = x - 1$.
   a. Find $(f + g)(x)$ and evaluate the sum when $x = 3$.
   b. Find $(f - g)(x)$ and evaluate the sum when $x = 3$.

4. Given $f(x) = -x + 1$ and $g(x) = x^2$,
   a. Find$(f \cdot g)(x)$ and then evaluate the product when $x = 2$.
   b. Find $\left(\frac{f}{g}\right)(x)$ and then evaluate the product when $x = 2$.
   c. Find $\left(\frac{g}{f}\right)(x)$ and then evaluate the product when $x = 2$.

Section 3	Types of functions

✿ Section Overview

Dear learner, in this section you will see three different types of functions. These are one -to- one function (Injective function)**,** onto function (Surjective function) **and** one- to- one correspondence (Bijective)

Section Objectives

By the end of this section, you will be able to: -

- Identify types of functions.
- Draw the graphs of different functions.

? What is a function?

A function is a kind of rule that, for one input, it gives you one output. An example of this would be $y = x + 1$. If you put in anything for x, you get one output for y. We would say that y is a function of x since x is the input value. Many widely used mathematical formulas are expressions of known functions. For example, the formula for the area of a circle, $A = \pi r^2$, gives the dependent variable A (the area) as a function of the independent variable r (the radius).

One-to-one function (Injective function)

? What is one-to-one function?

If each element in the domain of a function has a distinct image in the co-domain, then the function is said to be one -to- one function.
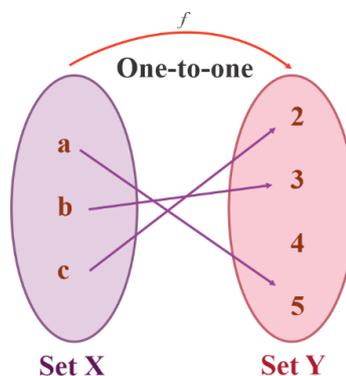
Figure 1.11

Definition 2.4

☞ A function $f: A \rightarrow B$ is called one-to-one if and only if for all $x_1, x_2 \in A$,

$f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Example 1

Let $f: R \rightarrow R$ given by $f(x) = 3x + 5$. Show that f is one-to-one.

Solution:

Let $x_1, x_2 \in R, f(x_1) = f(x_2)$ implies $3x_1 + 5 = 3x_2 + 5$ i.e., $x_1 = x_2$

Therefore, f is one-to-one.

Example 2

In figure 1.12, which of the following sets of values represent a one-to-one function?
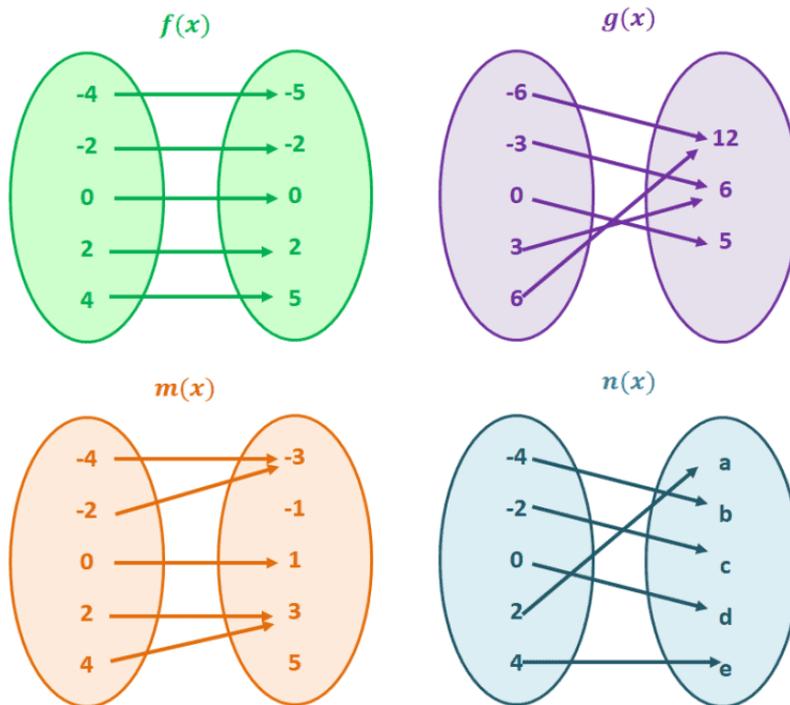
Figure 1.12

Solution:

For the first set $f(x)$, we can see that each element from the right side is paired up with a unique element from the left. Hence, $f(x)$ is a one-to-one function.

The set $g(x)$, shows a different number of elements on each side. This alone will tell us that the function is not a one-to-one function.

Some values from the left side correspond to the same element found on the right, so $m(x)$ is not a one-to-one function as well.

Each of the elements on the first set corresponds to a unique element on the next, so $n(x)$ represents a one-to-one function.

Onto-function (Surjective Function)

☞ A function is called an onto function if each element in the co-domain has at least one pre- image in the domain
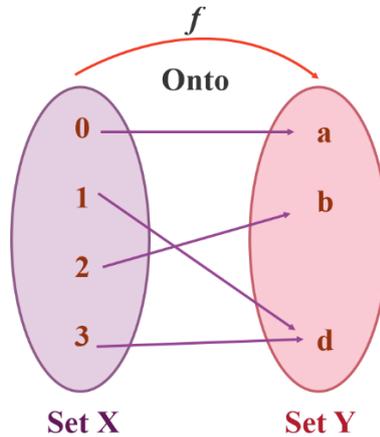
Figure 1.13

Example 1

Which of the following sets of values represents an injective and surjective function?

a.
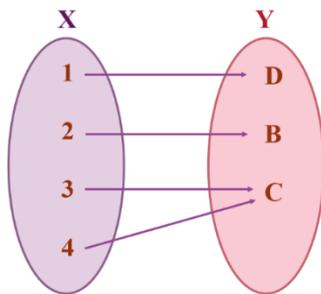
b.
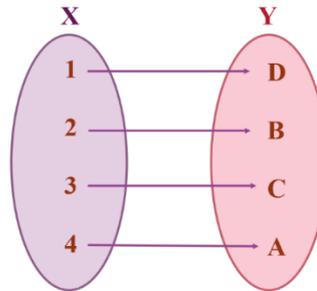
c.
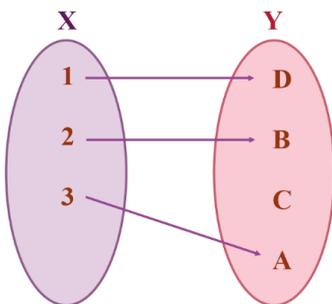
d.
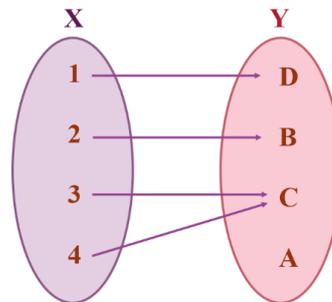
Figure 1.14

Solution:

a. Not one-to-one but it is an onto (surjective) function.

b. Both one-to-one and onto function.

c. An injective non-surjective function (injection).

d. A non-injective non-surjective function.

One to one correspondence (Bijective function)

👉 A function $f : A \to B$ is said to be a one-to-one correspondence if f is both one to one and onto.
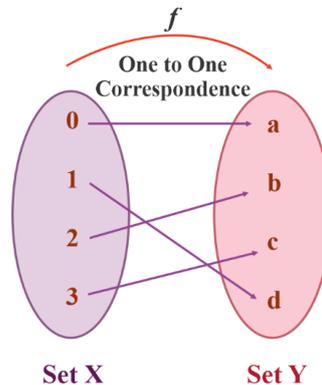
Figure 1.15

Example 1

The function $f(x) = x^2$ from the set of positive real numbers to the set of positive real numbers is both injective and surjective. Thus, it is also a bijective. Is it true that whenever $f(x) = f(y)$, $x = y$? Imagine $x = 3$, then $f(x) = 9$. Now I say that $f(y) = 9$, what is the value of y? It is 3, so $x = y$.

2.5 Graphs of functions

Dear learner, in this section, you will learn how to draw graphs of functions such as

$y = ax + b$ and $y = ax^2 + bx + c$ with special emphasis on linear and quadratic functions. You will study some of the important properties of graphs as in the following:

Definition 2.5

👉 If f is a function with domain A, then the graph of f is the set of all ordered pairs $\{(x, f(x)), x \in A\}$

That is, the graph of f is the set of all points (x, y) such that $f(x) = y$. This is the same as the graph of the equation $f(x) = y$ discussed in Cartesian coordinates. The graph of a function allows us to translate between algebra and pictures or geometry.

Graph of linear function:

> ### Definition 2.6
>
> ☞ If a and b are fixed real numbers, $a \neq 0$, then $f(x) = ax + b$ for every real number x is called a linear function. If $a = 0$, then $f(x) = b$ is called a constant function. Sometimes linear functions are written in the form $y = ax + b$.

Example 1

$f(x) = 2x + 1$ is a linear function with $a = 2$ and $b = 1$

Example 2

$f(x) = 2$ is a constant function.

To work on graphic functions, you can pick a few values of x and calculate the corresponding values of y or $f(x)$, plot the resulting points $(x, f(x))$ and connect the points.

Example 3

Draw the graph of $f(x) = 2$

Solution:

Construct a table for the value of the function; plot the ordered pairs and draw a line through the points to get the required graph.

Table 1.2

x	-3	-2	-1	0	1	2	3
$f(x)$	2	2	2	2	2	2	2
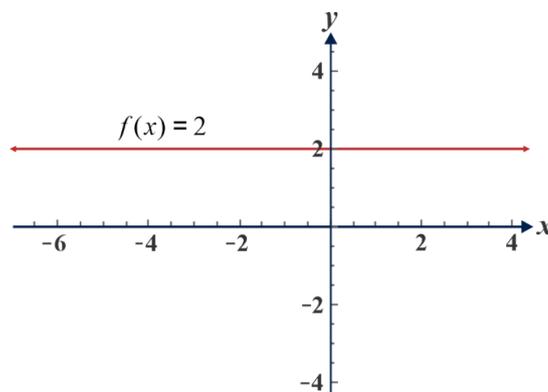
Figure 1.16 Graph of the constant function $f(x) = 2$

Example 4

Fill in the tables shown in Table 1.3 for the function $f(x) = x$ and draw its graph.

Table 1.3

x	−3	−2	−1	0	1	2	3
$f(x)$							

If $f(-3) = -3, f(-2) = -2, f(-1) = -1, f(0) = 0, f(1) = 1, \ f(2) = 2, f(3) = 3$.

So, the table becomes

Table 1.4

x	−3	−2	−1	0	1	2	3
$f(x)$	−3	−2	−1	0	1	2	3

When you plot the corresponding points on the Cartesian plane and connect the points to get a picture of the graph of function, the ordered pairs will give you a graph in the shape shown in Figure1.17. The domain is the set of all real number x and the range is the set of all real numbers y.
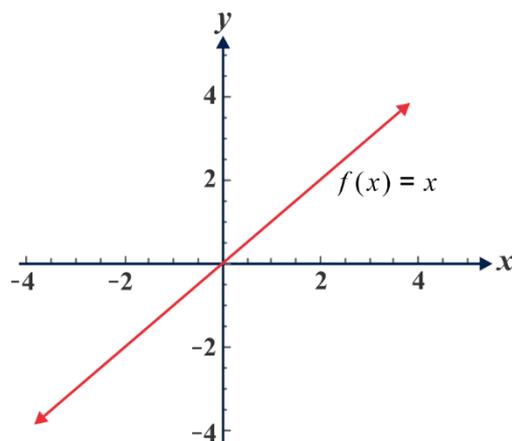
Figure 1.17 the graph of $f(x) = x$

Example 5

Consider the linear function $f(x) = x + 3$ and evaluate the values of the function for the x values in Table 1.5 and draw its graph.

Table 1.5

x	−4	−3	−2	−1	0	1	2
$f(x) = x + 3$							

If $x = -4, f(-4) = -1, f(-3) = 0, f(-2) = 1, f(-1) = 2, f(0) = 3, f(1) = 4, \ f(2) = 5,$ the table becomes

Table 1.6

x	-4	-3	-2	-1	0	1	2
$f(x) = x + 3$	-1	0	1	2	3	4	5

Table 1.6 is pairing the values of x and $f(x)$. This is taken as a representation of set of ordered pairs $(-4, -1), (-3, \ 0), (-2, 1), (-1, 2), (0, 3), (1, 4)$ and $(2, 5)$. Now you can plot these points in a coordinate system to draw the graph of the given function.
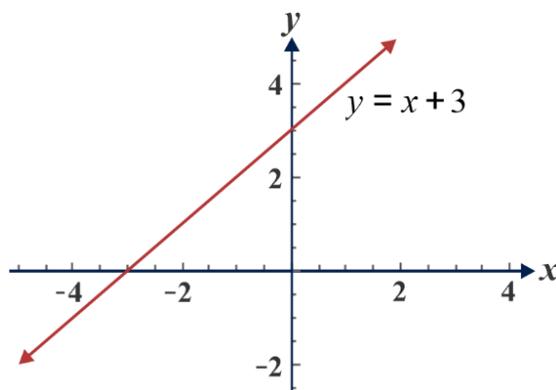
Figure 1.18 the graph of $f(x) = x + 3$

Example 6

Draw the graph of the linear function $f(x) = -2x + 6$.

Table 1.7

x	-2	-1	0	1	2	3
$f(x) = -2x + 6$	10	8	6	4	2	0
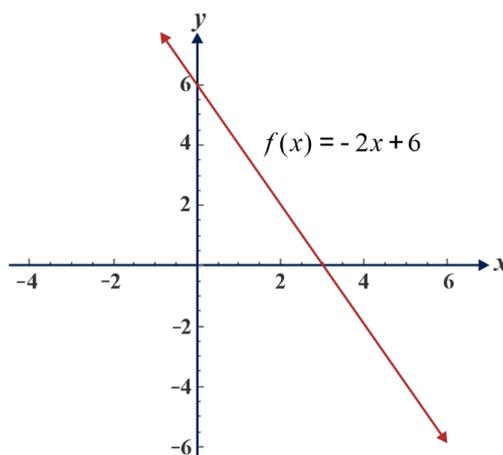
Figure 1.19 the graph of $f(x) = -2x + 6$

Example 7

Draw the graph of the following functions in the same Cartesian coordinate system.

a. $f(x) = x + 2$ **b.** $g(x) = -x + 2$ **c.** $h(x) = 2x + 2$ **d.** $k(x) = -2x + 2$

Solution:

We need two points to draw a line. However, we generally choose three, and the third point is a good check that we do **not** make a mistake.

a. $f(x) = x + 2: f(0) = 2,$
$f(1) = 3, f(2) = 4,$

b. $g(x) = -x + 2 : g(0) = 2,$
$g(1) = 1, g(2) = 0,$

c. $h(x) = 2x + 2 : h(0) = 2,$
$h(1) = 4, h(2) = 6,$

d. $k(x) = -2x + 2 : k(0) = 2,$
$k(1) = 0, k(2) = -2.$

$f(x) = x + 2 \quad h(x) = 2x + 2 \quad\quad k(x) = -2x + 2 \quad g(x) = -x + 2$
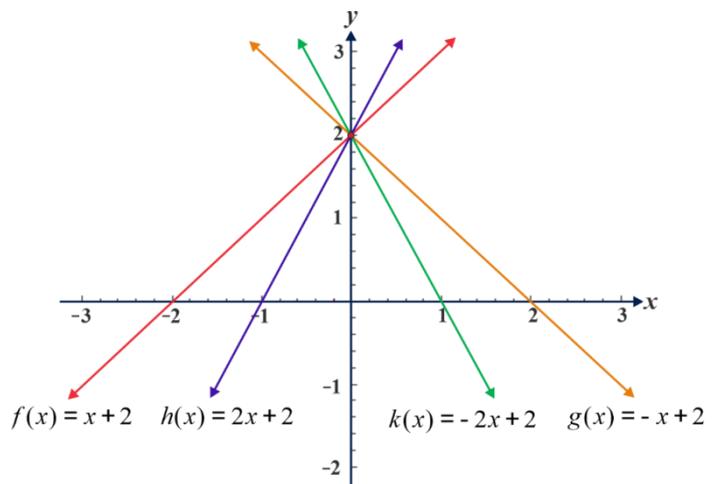
Figure 1.20 the graph of four functions f, g, h, k intersecting at a point $(0,2)$.

As indicated on Fig.1.25, the graphs of the four functions cross the y-axis at $y = 2$ because the value of b is always 2 for the equation $y = ax + b$ where b is the y-intercept. Thus, you can see that if $a > 0$ then the straight line goes up as x increases, and the bigger a gets the faster the line goes up. Similarly, if $a < 0$ then the line goes down as x increases, and the bigger a gets in absolute terms, the faster the line goes down.

From the graphs given above, you have noticed that:

a. Graphs of linear functions are straight lines.

b. If $a > 0$, then the graph of the linear function $f(x) = ax + b$ is increasing.

c. If $a < 0$, then the graph of the linear function $f(x) = ax + b$ is decreasing.

d. If $a = 0$, then the graph of the linear function $f(x) = b$ is a horizontal line.

Slope and intercept:

The slope indicates the steepness of a line and the intercept indicates the location where it intersects an axis. Linear functions are written in the form $y = ax + b$ and a is called its slope and b is its y-intercept.

Example 1

If $f(x) = 3x - 6$, then find the x and y-intercepts.

Solution:

To find x- intercept, put $f(x) = y = 0$ and then solve for x as:

$$0 = 3x - 6,$$

$$x = 2.$$

So, the x- intercept is $(2, 0)$.

To find y-intercept, put $x = 0$. And solve, you have $f(0) = -6$.

So, the y – intercept is $(0, -6)$.

Example 2

Draw the graph of the function $f(x) = 2x - 3$.

Solution:

Find the x and y- intercept.

The x- intercept is the ordered pair where $y = 0$ that

is $\left(\frac{3}{2}, 0\right)$. And the y- intercept is the ordered pair

where $x = 0$ that is $(0, -3)$. Plot these intercepts on a

coordinate system and draw a line that passes through

them.
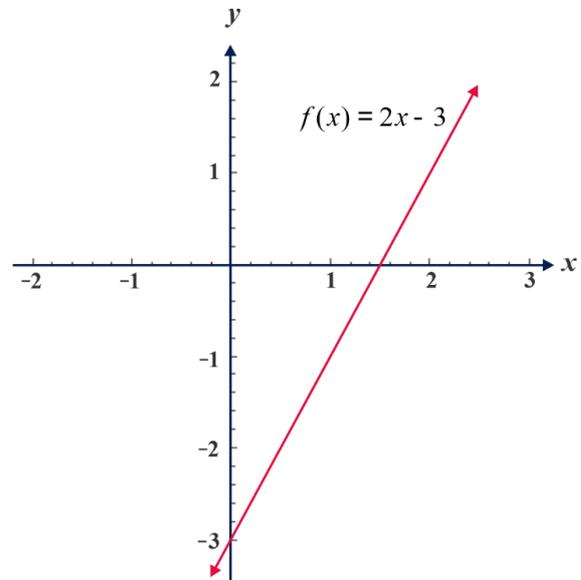
Figure 1.21 the graph of $f(x) = 2x - 3$

Example 3

If $f(x) = 2 - x$, is the graph of the function increasing or decreasing function?

Solution:

Since $f(x) = 2 - x$ is the same as $f(x) = -x + 2$ and the coefficient of x is -1, the graph the

function is decreasing.

Graphs of quadratic functions

> A function defined by $f(x) = ax^2 + bx + c$ where a, b, c are real numbers and
> $a \neq 0$ is called a quadratic function. The point a is the leading coefficient of f.

Example 1

$f(x) = 3x^2 - 2x + 5$ is a quadratic function with $a = 3$, $b = -2$ and $c = 5$.

Quadratic functions are useful in many applications in mathematics when a linear function is not

sufficient. For example, the motion of an object thrown either upward or downward is modeled by a

quadratic function. The graph of a quadratic function is a curve called a parabola. Parabolas may

open upward or downward and vary in" width" or" steepness", but they all have the same basic" U"

shape. All parabolas are symmetric with respect to a line called the axis of symmetry. A parabola intersects its axis of symmetry at a point called the vertex of the parabola.

Many quadratic functions can be graphed easily by hand using the techniques of stretching/ shrinking and shifting (translation) the parabola $y = x^2$.

Activity 1.7

a. Make a table of ordered pairs that satisfy the function $f(x) = x^2$.

b. Find the x and y intercepts of f.

c. Plot the points (x, x^2) on xy-coordinate system.

d. Find the domain and range of f.

Have you done it? Well!

The following are observations of the graph of $h(x) = x^2$.

➢ Since squaring any number gives a positive number, the values of y are all positive, except when $x = 0$, in which case $y = 0$.

➢ As x increases in size, so does x^2 but the increase in the value of x^2 is 'faster' than the increase in x.

➢ The graph of $y = x^2$ is symmetric about the y-axis $(x = 0)$. For example, if $x = 3$, the corresponding y value is $3^2 = 9. If x = -3$, then the y value is $(-3)^2 = 9$. The two x - values are equidistant from the y-axis: one to the left and one to the right, but the two $y\ values$ are in the same height above the x- axis.
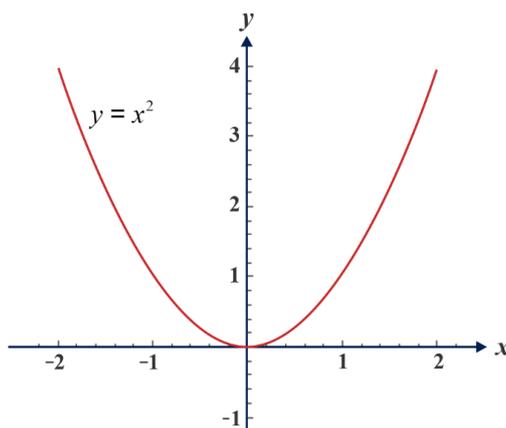
Figure 1.22 the graph of $y = x^2$

Referring to figure1.22, we observe the following

➢ the line $x = 0$ (y-axis) is called the line of symmetry for this quadratic function.

➢ the line $y = 0$ (x-axis) is called the orthogonal axis for this quadratic function.

If the equation is, say, $y = x^2$ then the graph will be similar to that of $y = 2x^2$ but will lie above it.

For example, when $x = 1$ the value of x^2 is 1, but the value of $2x^2$ is 2. The y value for $y = 2x^2$

is above that of $y = x^2$. Similarly, for the equation $y = \frac{x^2}{2}$, the graph is similar to that of $y = x^2$ except lying below.
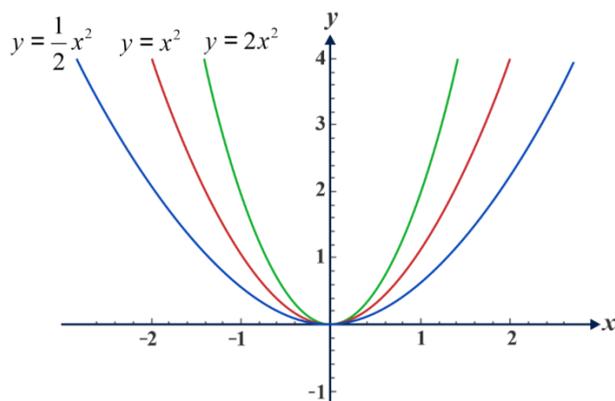
Figure 1.23 the graph of $y = ax^2, a = \frac{1}{2}, 1, \ 2$

Sketch of $y = ax^2$ for different positive values of a are shown in Figure 1.23.

Consider now the choice $a = -1$, with the equation $y = -x^2$. In this case the graph of the equation will have the same shape but now, instead of being above the x -axis it is below. When $x = 1$ the corresponding y value is -1.

Table 1.8

x	-3	-2	-1	0	1	2	3
$f(x) = -x^2$	-9	-4	-1	0	-1	-4	-9
$g(x) = -2x^2$	-18	-8	-2	0	-2	-8	-18
$h(x) = -\frac{1}{2}x^2$	$-\frac{9}{2}$	-2	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-2	$-\frac{9}{2}$

Examples $of\ y = ax^2$ for various negative values of a are sketched in figure 1.24.
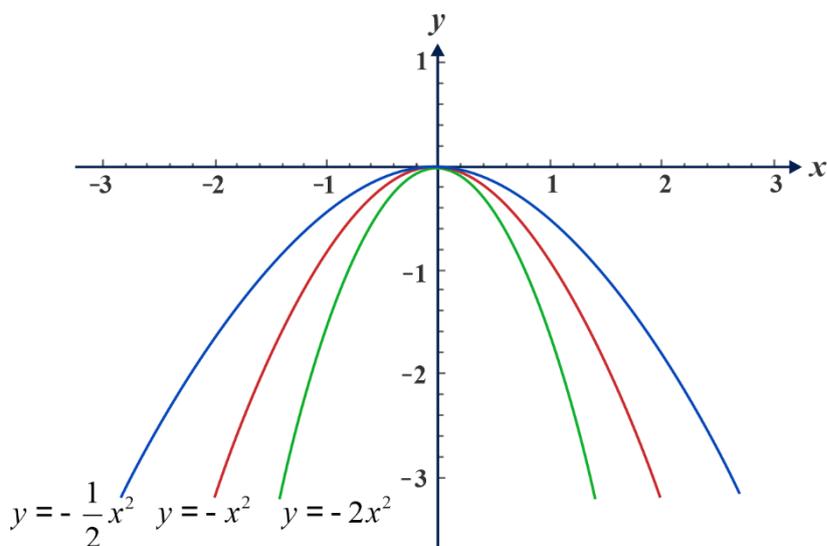
Figure 1.24 the graph of $y = ax^2, a = -2, \ -\frac{1}{2}, -1$

Graph of $y = ax^2 + c$.

This type of quadratic function is similar to the basic ones of the previous functions discussed but with a constant c added in the function $y = ax^2$, i.e., having the general form

$y = ax^2 + c$. As an example of this, $y = x^2 + 2$. Comparing this with the function $y = x^2$, the only difference is the addition of 2 units. When $x = 1$, $x^2 = 1$, but $x^2 + 2 = 1 + 2 = 3$. When $x = 2, x^2 = 4$, but $x^2 + 2 = 4 + 2 = 6$.

Figure 1.25 graph of $y = x^2 + c$, for $c = 2, 0, -2$

That is the values of $y = x^2$ have been lifted by 2 units. This happens for all of the x values so the shape of the graph is unchanged but it is lifted by 2 units. Similarly, the graph of $y = x^2 - 2$ will be lowered by 2units.

Graph of $y = a(x - k)^2$

In the examples considered so far, the axis of symmetry is the y-axis, i.e., the line $x = 0$. The next possibility is a quadratic function which has its axis of symmetry not on the y-axis. A case in point to this function:

$y = (x - 3)^2$ has the same shape and the same orthogonal axis as $y = x^2$ but the axis of symmetry is the line $x = 3$. i.e., shift to the right by 3 units. The points $x = 0$ and $x = 6$ are equidistant from 3.

When $x = 0$ the y value is $(0 - 3)^2 = 9$. When $x = 6$ the y value is $(6 - 3)^2 = 9$.

The points on the curve at these values are both 9 units above the x- axis.

This is true for all numbers which are equidistant from 3.

Figure 1.26 the graph of $y = (x - 3)^2$

From the graph of quadratic functions of the form $f(x) = ax^2$, $y = ax^2 + c$, $a \neq 0$, c is any real number, we can summarize:

1. If $a > 0$, the graph opens upward and if $a < 0$, the graph opens downward.

2. The vertex is $(0,0)$ for $f(x) = ax^2$ and $(0, c)$ for $y = ax^2 + c$.

3. The domain is all real numbers.

4. The vertical line that passes through the vertex is the axis of the parabola (or the axis of symmetry).

5. If $a > 0$, the range is the set of non- negative real number y for $f(x) = ax^2$ and the set of real number such that $y \geq c$ for $y = ax^2 + c$

6. If $a < 0$, the range is the set of non- positive real number y for $f(x) = ax^2$ and the set of real number such that $y \leq c$ for $y = ax^2 + c$

Graphs of $y = a(x - k)^2 + m \ (a < 0)$.

So far, two separate cases have been discussed: first a standard quadratic function has its orthogonal axis shifted up or down; second a standard quadratic function has its axis of symmetry shifted left or right. The next step is to consider quadratic functions that incorporate all shifts.

Example 1

The quadratic function $y = x^2$ is shifted so that its axis of symmetry is at $x = 3$ and its orthogonal axis is is at $y = 2$.

   a. Write down the equation of the new curve.

   b. Find the coordinates of the point where it crosses the y-axis.

   c. Sketch the new curve shifted from $y = x^2$

Solution:

   a. The new curve $y = (x - 3)^2$ is symmetric about $x = 3$ and is shifted up by 2 units, so its equation is $y = (x - 3)^2 + 2$.

   b. The curve crosses the y-axis, when $x = 0$.

      Putting this into the equation

      $y = (x - 3)^2 + 2$, the corresponding value

      of y is $y = (0 - 3)^2 + 2 = 11$,

      so, the curve crosses the y −axis at $y = 11$.

   c. The curve is shown in figure1.27.
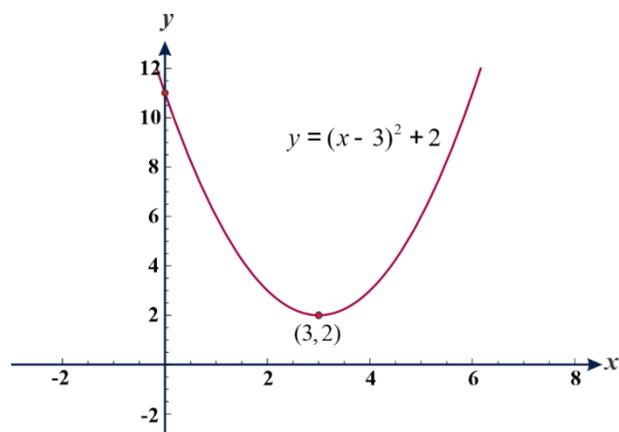
Figure 1.27 the graph of $y = (x - 3)^2 + 2$

Note that the vertex of the graph of $f(x) = (x + k)^2 + c$ is $(-k, c)$ and the vertex of the graph of $f(x) = (x - k)^2 - c$ is $(k, -c)$. On the contrary, the vertex of the graph of $f(x) = (x + k)^2 - c$ is $(-k, -c)$ and the vertex of the graph of $f(x) = (x - k)^2 + c$ is (k, c).

The final section is about sketching general quadratic functions, i.e., one of the forms

$y = ax^2 + bx + c$.

The algebraic expression must be rearranged so that the line of symmetry and the orthogonal axis can be determined. The procedure required is completing the square.

Example 2

A quadratic function is given by $y = x^2 + 6x + 11$.

    a. Apply completing the square method for the function.

    b. Use this to determine the axis of symmetry and the orthogonal axis of the curve.

    c. Find the points on the x and y axes where the curve crosses them.

    d. Sketch the function.

Solution:

    a. Completing the square:

$$y = x^2 + 6x + 11$$
$$= x^2 + 6x + 9 - 9 + 11$$
$$= (x + 3)^2 + 2$$

    b. This is the function $y = x^2$ moved to the left so that its axis of symmetry is $x = -3$ and shifted up by 2, i.e., its orthogonal axis is $y = 2$.

    c. The function is $y = (x + 3)^2 + 2$. It will not cross the x-axis, i.e., the graph has not x-intercept. Putting $x = 0$ into the original form of the function $y = x^2 + 6x + 11$, gives $y = 11$, i.e., it crosses the y-axis at $y = 11$.

    d. The function is shown in figure1.28.
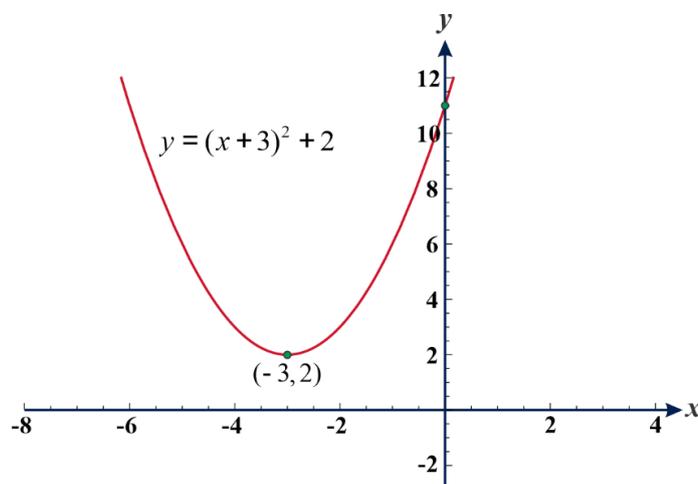
Figure 1.28 the graph of the function $y = x^2 + 6x + 11$

Graphs of $y = a(x - k)^2 + m \; (a < 0).$

■ **Example 1**

A quadratic function is given as $y = -x^2 + 2x + 3.$

    a. Complete the square on this function.

    b. Use this to determine the axis of symmetry and the orthogonal axis of the curve.

    c. Find the points on the x and y axes where the curve crosses them.

    d. Sketch the function.

Solution:

    a. $\quad y = -x^2 + 2x + 3$

$$= -(x^2 - 2x) + 3 = -((x - 1)^2 - 1) + 3$$

$$= -(x - 1)^2 + 1 + 3$$

$$y = -(x - 1)^2 + 4$$

    b. This is the function $y = -x^2$ moved to right so that its axis of symmetry is $x = 1$ and shifted up by 4, i.e., its orthogonal axis is $y = 4.$

    c. The function is $y = -(x - 1)^2 + 4.$ This will cross the x-axis, when $y = 0$

      i.e., when $-(x - 1)^2 + 4 = 0$

$$4 = (x - 1)^2$$

    taking square roots yields $x - 1 = \pm 2$ implies $x = 1 \pm 2$, i.e., $x = -1 \; or \; 3.$

    Putting $x = 0$ into the original form of the function at the $y = -x^2 + 2x + 3.$

    Gives $y = 3$, i.e., it crosses the y-axis at $y = 3.$

    d. The function is shown in figure1.29.
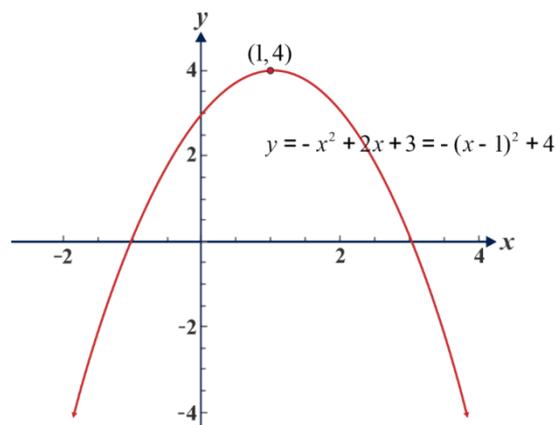
Figure 1.29 the graph of $y = -x^2 + 2x + 3$

Note

1. The graph of $f(x) = (x + k)^2 + c$ opens upward.

2. The graph of $f(x) = -(x + k)^2 + c$ opens downward.

3. The vertex of the graph of $f(x) = (x + k)^2 + c$ is $(-k, c)$ and the vertex of the graph of

   $f(x) = (x - k)^2 - c$ is $(k, -c)$. Similarly, the vertex of the graph of

   $f(x) = (x + k)^2 - c$ is $(-k, -c)$ and the vertex of the graph of $f(x) = (x - k)^2 + c$ is (k, c).

Checklist

Dear learner, now it is time to check your understanding about relation. Read each question and put a tick (" ✓ ") if you can do the task and an "x" if you can't on the space provided.

- Can you define a function? ☐
- Can you give two examples' functions? ☐
- Can you define one-to-one, onto and one-to-one correspondence function? ☐
- can you list types of functions? ☐
- Can you draw graphs of quadratic functions? ☐
- Can you find the domain of the functions? ☐
- Can you find the range of the functions? ☐

If your response to the above check list questions is positive, proceed to do the self-test exercise given below and check your answer at the end of this module. If not please review the section again and again till you think you do not have any problem.

Self-Test Exercise 3

1. Which of the following mappings are injective, surjective, bijective function?
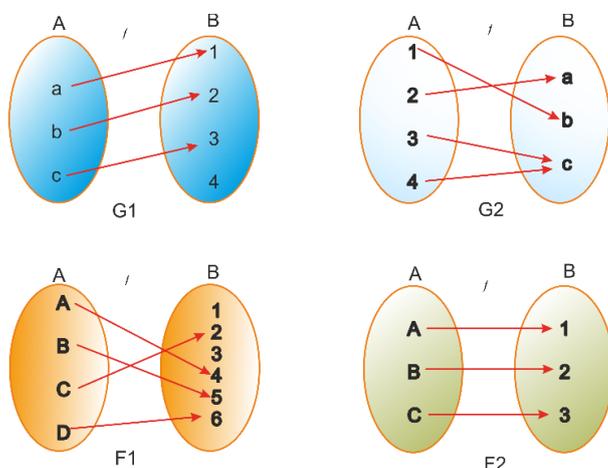
Figure 1.30

2. Which of the following are one-to-one functions?

   a. A is the set of ordered pair (x, y): $y = |x - 4|$.

   b. $B = f: \Re \rightarrow \Re$ given by $f(x) = x + 3$.

3. Which of the following functions are onto?

   a. $f: \Re \rightarrow \Re, f(x) = x + 7$

   b. $g: [0, \infty) \rightarrow \Re, g(x) = x^2$

4. Identify if the following function is an injective, surjective, and/or bijective function?

   a. $f: R \rightarrow R, f(x) = x - 1$ **b.** $f: R \rightarrow R, f(x) = -3x$

5. Construct tables of values of the following functions for the given domains:

   a. $f(x) = 3x - 1$; $x = -3, -2, -1, 0, 1, 2, 3$

   b. $f(x) = -2x + 3$; $x = -1, -0.5, 1, 2, 3, 4$

   c. $f(x) = 4 - 2x$; $x = -1, 0, 1, 2, 3, 4$

   d. $f(x) = \frac{x}{2} - 1$; $x = -8, -4, -2, 0, 2, 4$

6. Draw the graph of the following functions.

   a. $f(x) = 1 - x^2$ **b.** $g(x) = x^2 - 3$

   c. $h(x) = 2x^2 - 2$ **d.** $k(x) = (x - 2)^2 + 1$

Section 4   Applications of Relations and Functions

Applications involving relations

❋ Section Overview

Dear learner, in this section you will see application of relations and of functions.

> ### Section Objectives
>
> **By the end of this section, you will be able to: -**
>
> - solve Quadratic inequality.
> - find the Maximum and Minimum values of the functions.

Example 1

The data in table1.9 depicts the length of a woman's femur and her corresponding height. Based on these data, a forensics specialist can find a linear relationship between heights y and femur x cm:

$y = 0.906x + 24.3, 40 \leq x \leq 55$.

Table 1.9

Length of femur (cm) x	Height (inches) y
45.5	65.5
48.2	68.0
41.8	62.2
46.0	66.0
50.4	70.5

From this type of relation, the height of a woman can be inferred based on skeletal remains.

    a. Find the height of the woman whose femur is 46.0 cm.

    b. Find the height of the woman whose femur is 51.0 cm.

Solution:

    a. $y = 0.906x + 24.3$

         $= 0.906(46.0) + 24.3\ = 65.976$

    The woman is approximately 66.0 inches tall.

    b. $y = 0.906x + 24.3$

         $= 0.906(51.0) + 24.3\ = 70.506$

    The woman is approximately 75.5 inches tall.

Example 2

If the equation $x^2 + 18x + 81$ represents the area of the square, what is the perimeter of the square if $x = 10$?

Solution:

$x^2 + 18x + 81$ factors into $(x + 9)(x + 9)$

Since this represents the area of a square where length = width, then

$(x + 9) =$ length and $(x + 9) =$ width

Perimeter of a square $= 2$(length + width) $= 2(x\ +\ 9\ +\ x + 9) = 2(2x + 18) = 4x + 36$

For $x = 10,$ Perimeter $4(10) + 36 = 40 + 36 = 76$

Example 3

The width of a square is 1 less than twice its width. What is its length?

Solution:

When 1 less than a number, the algebraic symbol is is $x - 1$

Let the width be x

when x is 1 less than $2x$

$1 = 2x - x$ and $x = 1$. In a square, length = width. So, the length is 1.

Minimum and Maximum values of quadratic functions

Dear learner, some of the application of functions will be treated like minimum and maximum value, solving inequality by graphical method.

Suppose you throw a stone upward. The stone turns down after it reaches its maximum height. Similarly, a parabola turns after it reaches a maximum or a minimum y value.

Activity 1.8

Let $f(x)$ be a quadratic function. Do you know how to determine the maximum or minimum value of $f(x)$.

Example 1

The minimum value of a quadratic function expressed as $f(x) = (x + k)^2 + c$ is c.

Similarly, the maximum value of $f(x) = -(x + k)^2 + c$ is c.

Example 2

Find the maximum value of the function $f(x) = -x^2 + 6x - 8$, and sketch its graph.

Solution:

$f(x) = -x^2 + 6x - 9 + 9 - 8$

$\qquad = -(x^2 - 6x + 9) + 1;$

$f(x) = -(x - 3)^2 + 1.$

The graph of $f(x) = -(x - 3)^2 + 1$ has vertex $(3, 1)$ and hence the maximum value of f is 1.

In this case, the range of the function is $\{y: y \leq 1\} = (-\infty, 1]$
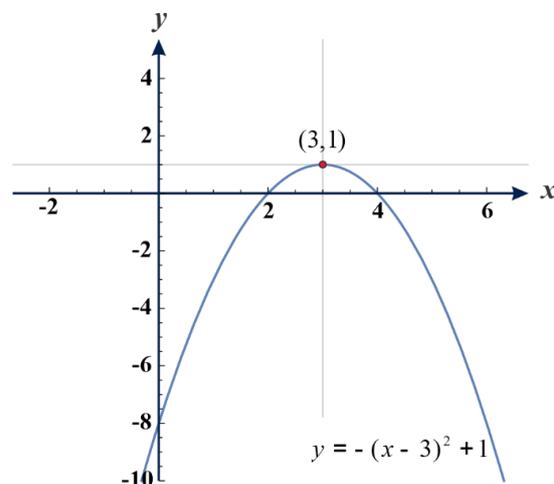
Figure 1.31 the graph of the function $f(x) = -(x - 3)^2 + 1$

Graphical method of solving quadratic equations

Activity 1.9

For the quadratic function: $y = x^2$, complete the table, and plot the pair of coordinates on the coordinate plane to draw the graph.

Table 1.10

x	...	-3	-2	-1	0	1	2	3	...
y	...	9	4	1					

In general, the graph of quadratic function has u-shaped curve and one extreme point, called vertex. In the Activity, the vertex of the function is $(0,0)$. In order to find the solutions of a quadratic equation using a graph:

1. Rearrange the equation so that one side is equal to zero (if necessary).
2. Draw the graph of the quadratic function.
3. Read off the x-coordinate(s) of the point(s) where the curve crosses the x-axis.

The quadratic function is transformed using completing the square method as follows:

$$y = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - a \cdot \frac{b^2}{4a^2} = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

Hence, the vertex of the function is $(-\frac{b}{2a}, \frac{4ac - b^2}{4a})$.

(That is, translate $y = ax^2$ by $-\frac{b}{2a}$ in x-axis direction, and by $\frac{4ac - b^2}{4a}$ in y-axis direction.

It uses the vertex formula $\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$ to get the vertex which gives an idea of what values to choose to plot the points. If the graph of the quadratic function crosses the x-axis at two points then we have two solutions. The x-intercept(s) of a graph is/are the solution(s) of the equation. If the graph touches the x-axis at one point then we have one solution. If the graph does not intersect with the x-axis then the equation has no real solution.

Example 1

Find the solutions of the equation $x^2 - 4 = 0$ graphically.

Solution:

Draw the graph of the quadratic function. $y = x^2 - 4$

Read off the x-coordinate(s) of the point(s)

where the curve crosses the x-axis.

The roots are $x = -2$ and $x = 2$

These are the solutions of $x^2 - 4 = 0$
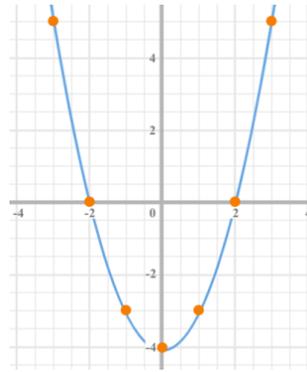
Figure 1.32 the graph of the function $\quad y = x^2 - 4$

Example 2

Solve $x^2 - 2x - 3 = 0$ graphically.

Solution:

The solution for the equation $x^2 - 2x - 3 = 0$ can be obtained by looking at the points where the

graph $y = x^2 - 2x - 3$ cuts at the x-axis (i.e. $y = 0$).

$\quad y = x^2 - 2x - 3 = (x - 1)^2 - 4$, hence the vertex is $(1, -4)$

In addition to the vertex, plot some points, such as y-intercept $(0, -3)$ and other

points $(-1, 0), (2, -3), (3, 0)$ to draw graph.

The graph $y = x^2 - 2x - 3$ cuts the x-axis at $x = -1$ and $x = 3$. So, the solution for the equation is

$x = -1$ and $x = 3$.
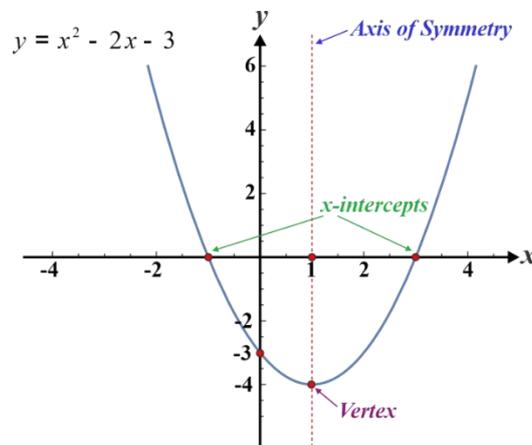
Figure 1.33 the graph of the function $\quad y = x^2 - 2x - 3$.

Example 3

Solve $-x^2 + 6x - 9 = 0$ graphically.

Solution:

The solution for the equation $-x^2 + 6x - 9 = 0$ can be obtained by looking at the points where the

graph $y = x^2 + 6x - 9$ cuts at the x-axis (i.e. $y = 0$).

$\quad y = -x^2 + 6x - 9 = -(x - 3)^2$, hence the vertex is $(3, 0)$

40

Plot other points, y-intercept $(0, -9)$, and $(1, -4), (2, -1), (4, -1)$ to draw graph.

The graph $y = -x^2 + 6x - 9 = 0$ cuts the x-axis only at $x = 3$. So, the solution for the equation is $x = 3$.
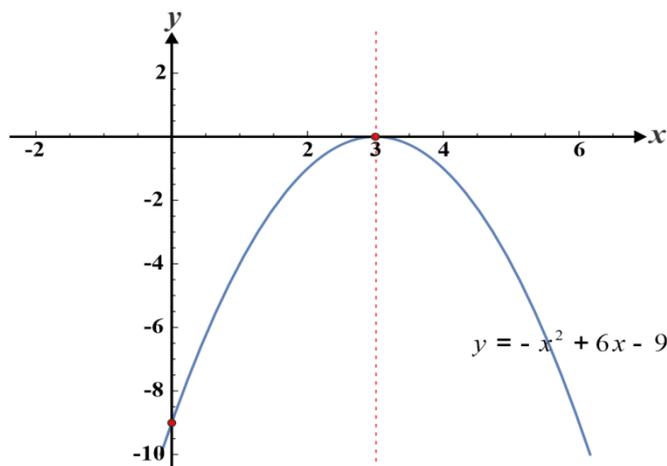
Figure 1.34 the graph of the function $y = -x^2 + 6x - 9$

Example 4

Solve $x^2 + 4x + 8 = 0$ graphically.

Solution:

The vertex for $y = x^2 + 4x + 8$ is given by using completing the square method as follows:
$$y = x^2 + 4x + 8 = (x + 2)^2 + 4$$

Then the vertex is (-2, 4). Furthermore, coefficient of x^2 is positive. Therefore, there is no x-intercept and hence, no real solution to the equation $x^2 + 4x + 8 = 0$
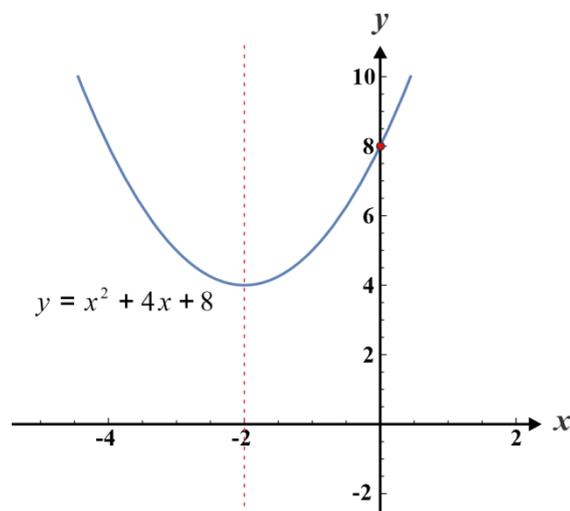
Figure 1.35 the graph of the function $y = x^2 + 4x + 8$

Note

In the graphs of quadratic functions, the sign on the coefficient a affects whether the graph opens up or down. If $a > 0$, then the graph opens up and if $a < 0$, then the graph opens down.

Graphical method of solving quadratic inequalities (1)

Activity 1.10

Given the following inequalities:

a. $y > x^2$ b. $y < x^2$ c. $y > x^2 + 4$

d. $y \geq 1 - x^2$ e. $y \leq x^2 + 2x$ f. $y \geq x^2 - 4$

1. Find the x-intercept and y-intercept of their respective equation.

2. Which region (inside or outside the parabola satisfy the inequality)?

In solving a quadratic inequality graphically, the following steps shall be followed.

1. Write the quadratic inequality in standard form $x^2 + bx + c \geq 0 \ \ or \ \ ax^2 + bx + c \leq 0$.

2. Graph the function $f(x) = ax^2 + bx + c$ using properties or transformations.

3. Determine the solution from the graph.

The graph of a quadratic function $f(x) = ax^2 + bx + c$ is a parabola. When we ask $ax^2 + bx + c < 0, \ f(x) < 0$ (Figure 1.36). We want to know when the parabola is below the x-axis. When we ask $ax^2 + bx + c > 0$, we are asking when $f(x) > 0$ (Figure 1.37). We want to know when the parabola is above the x-axis.
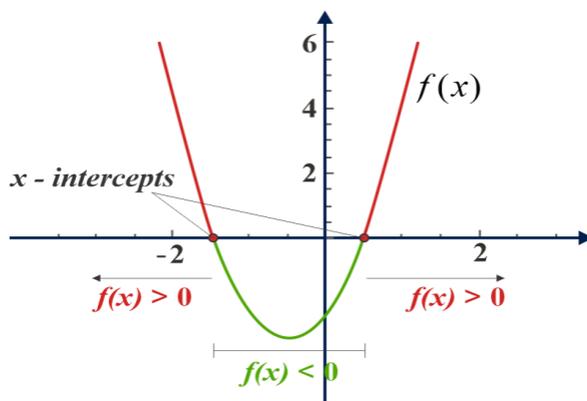
Figure 1.36 the graph of the function $y = ax^2 + bx + c$ for $a > 0$
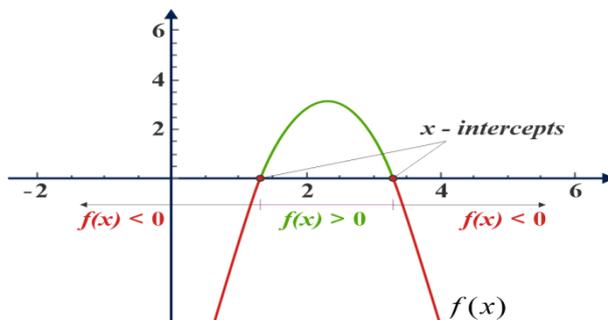
Figure 1.37 the graph of the function $y = ax^2 + bx + c$ for $a < 0$

Example 1

Solve $x^2 - 6x + 8 < 0$

Solution:

Step 1: Write the quadratic inequality in standard form. The inequality is in standard form.

Step 2: Graph the function $f(x) = ax^2 + bx + c$ using properties or transformations.

We will graph using the properties.

Look at a in the equation $f(x) = x^2 - 6x + 8$. $a = 1, b = -6,$ and $c = 8$. Since a is positive, the parabola opens upward.

$f(x) = x^2 - 6x + 8 = (x - 3)^2 - 1$. The vertex is $(3, -1)$. Using table to find other points.

x	1	2	3	4	5
$f(x) = x^2 - 6x + 8$	3	0	-1	0	3

The x-intercepts are $(2,0)$ and $(4, 0)$. We graph the x-intercepts, and the y-intercept. We connect these points to sketch the parabola (Figure 1.38).
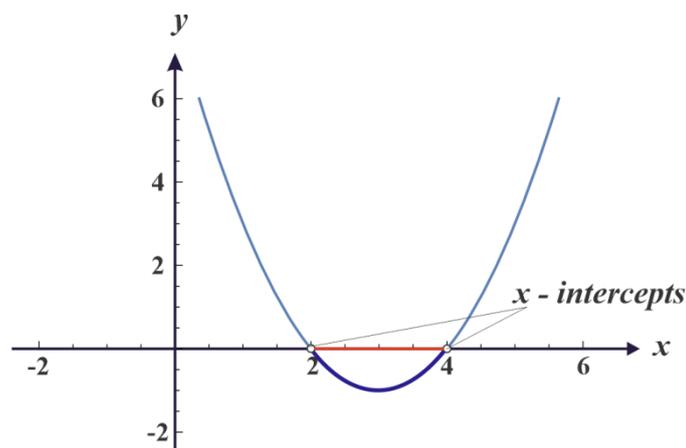
Figure 1.38 the graph of the function $y = x^2 - 6x + 8$

Step 3: Determine the solution from the graph.

The inequality asks for the values of x which make the function less than 0. Which values of x make the parabola below the x-axis? We do not include the values 2, 4 as the inequality is less than only.

The solution, in interval notation is $(2, 4)$ or $\{x: 2 < x < 4\}$ for the quadratic inequality $x^2 - 6x + 8 < 0$.

Graphical method of solving quadratic inequalities (2)

Example 2

Solve $-x^2 - 8x - 12 \leq 0$ graphically. Write the solution in interval notation.

Solution:

Consider the equation $f(x) = -x^2 - 8x - 12$. $a = -1, b = -8$, and $c = -12$. Since a is negative, the parabola opens downward.

We find $f(0) = -12$. Hence $(0, -12)$ is the y-intercept.

$$f(x) = -x^2 - 8x - 12 = -(x^2 + 8x + 12)$$
$$= -(x + 4)^2 + 4.$$

The vertex is $(-4, \ 4)$. Find other points using the table.

x	-6	-5	-4	-3	-2
$f(x) = -x^2 - 8x - 12$	0	3	4	3	0

So, the x-intercepts are $(-6, 0)$ and $(-2, 0)$. We graph the vertex, intercepts. We connect these points to sketch the parabola (Figure 1.39).
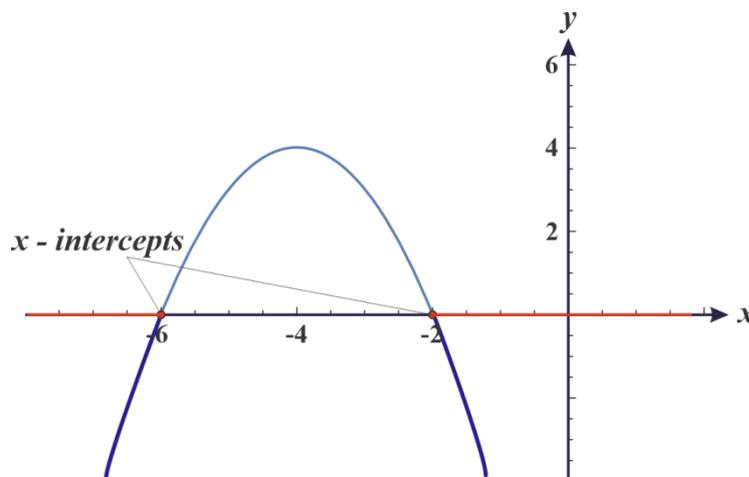
Figure 1.39 the graph of the function $\quad y = -x^2 - 8x - 12 \leq 0$.

The inequality asks for the values of x which make the function less than or equal to 0. We include the values -6 and -2 as the inequality is less than or equal to 0. The solution, in interval notation is $(-\infty, -6] \cup [-2, \infty)$ or $\{x : x \leq -6 \ \ or \ \ x \geq -2\}$ for the quadratic inequality $-x^2 - 8x - 12 \leq 0$.

Checklist

Dear learner, now it is time to check your understanding about relation. Read each question and put a tick (" $\checkmark$ ") if you can do the task and an "x" if you can't on the space provided.

- Can you define a function? ☐
- Can you define one-to-one, onto and one-to-one correspondence function? ☐
- Can you list types of functions? ☐
- Can you draw graphs of quadratic functions? ☐
- Can you find the domain of the functions? ☐
- Can you find the range of the functions? ☐

- Can you give two examples of functions? □

If your response to the above check list questions is positive, proceed to do the self-test exercise given below and check your answer at the end of this module. If not please review the section again and again till you think you do not have any problem.

Self-Test Exercise 4

1. There is 8m wire. With this wire, make a rectangle with length x m on one side.
   a. Express the width using x.
   b. Express the area S of the rectangle using x.
   c. If $f(x) = S$, draw the graph $f(x)$.
   d. Determine the domain and range of $f(x)$.

2. There is an isosceles triangle with height 6cm.
   a. If the base is x cm and express the area S using x
   b. Determine the domain and range of $f(x)$.

3. Find the vertex and the axis of symmetry of the following functions
   a. $f(x) = (x+3)^2 - 1$
   b. $f(x) = x^2 + 7x + 6$

4. Determine the minimum or the maximum value of each of the following functions and draw the graphs:
   a. $f(x) = x^2 - 2x + 2$ b. $f(x) = -x^2 - 6x - 8$

5. A metal wire 40 cm long is cut into two and each piece is bent to form a square. If the sum of their areas is 58 sq.cm, how long is each piece?

6. Use graphical method to solve the following.
   a. $x^2 - 1 = 0$ b. $x^2 + 2x + 1 = 0$

7. Solve the following quadratic inequalities graphically. Use the previous methods to check the answers.
   a. $x^2 + 6x + 5 < 0$ b. $x^2 + 6x - 7 \geq 0$

Key terms

✓ axis(orthogonal-axis)   ✓ leading coefficient   ✓ vertex

✓ axis of symmetry   ✓ linear function   ✓ x-intercept

✓ combination of functions   ✓ quadratic function   ✓ y-intercept

✓ constant function   ✓ relation   ✓ parabola

✓ coordinate system   ✓ range   ✓ turning point

✓ domain   ✓ slope   ✓ Function

Unit Summary

1. A Cartesian coordinate system in two dimensions (also called a rectangula**r coordinate** system) is defined by an ordered pair of perpendicular lines (axes), a single unit of length for both axes, and an orientation for each axis.

2. In relation, two things are related to each other by a relating phrase.

3. A relation is the set of ordered pairs.

4. The set of first components in the ordered pairs is called the **domain** of the relation. The set of second components in the ordered pairs is called the **range** of the relation.

5. A function is a special type of relation in which no two distinct ordered pairs have the same first element.

6. A function from $A\ to\ B$ can sometimes be denoted as $f: A \rightarrow B$, where the domain of the function f is A and the range of the function f is included in B, in which we say B contains the image of the elements of A under the function of f.

7. A numerical relation R is a function if and only if no **vertical line** in the plane intersects the graph of R in more than one point.

8. If each element in the domain of a function has a distinct image in the co-domain, then function is said to be one -to- one function.

9. A function is called an onto function if each element in the co-domain has at least one pre – image in the domain.

10. A function $f: A \rightarrow B$ is said to be a one- to-one correspondence if f is both one to one and onto.

11. Let f and g be two functions with overlapping domains. We define the sum, difference, product, and quotient as:

$$(f + g)(x) = f(x) + g(x) \qquad (fg)(x) = f(x)g(x)$$
$$(f - g)(x) = f(x) - g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$$

12. If a and b are fixed real numbers, $a \neq 0$, then $f(x) = ax + b$ for every real number x is called a linear function.

13. In $f(x) = ax + b$ for $a \neq 0, x \in \mathbb{R}$, a represent the slope, $(0, b)$ represents the y- intercept and $\left(-\frac{b}{a}, 0\right)$ represent the x- intercept.

14. A function defined by $f(x) = ax^2 + bx + c$ where a, b, c are real numbers and $a \neq 0$ is called a quadratic function. The point a is the leading coefficient of f.

15. We can sketch the graph of a linear function by using either a table of values, or the x and $y-$ intercepts.

16. We can sketch the graph of a quadratic function by using either a table of values, or shifting rule.

17. The graph of $f(x) = ax^2 + bx + c$ opens upward if $a > 0$ and downward if $a < 0$.

18. The vertex is the point on the coordinate system at which a graph of a quadratic function turns either upward or downward.

19. The axis of symmetry is a vertical line that passes through the vertex of the parabola.

20. The domain and range of linear functions are the set of real numbers.

21. The domain of a quadratic function $f(x) = ax^2 + c$ is the set of real numbers, whereas the range is;

    $\{y : y \geq c\}$ if the leading coefficient is positive and c is the value of y at the vertex.

    $\{y : y \leq c\}$ if the leading coefficient is negative and c is the value of y at the vertex.

    $For\ f(x) = ax^2 + bx + c$. In this case, $f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{(b^2 - 4ac)}{4a}$. The coordinate of the vertex is $\left(-\frac{b}{2a}, -\frac{(b^2 - 4ac)}{4a}\right)$. If $a > 0$, the range is $\left\{y : y \geq \frac{b^2 - 4ac}{4a}\right\}$.

    If $a < 0$, the range is $\left\{y : y \leq \frac{b^2 - 4ac}{4a}\right\}$.

Unit-1 Review Exercises

Part-I Write true if the statement is correct and false if the statement is incorrect.

1. Every function is a relation.

2. The range of a quadratic function $f(x) = ax^2 + c$ is the set of real numbers.

3. The maximum value of $f(x) = -(x + k)^2 + c$ is c.

4. If $f(x) = \sqrt{x + 1} + 2$ and $g(x) = \sqrt[3]{x}$, then $(f.g)(-1) = -2$.

5. Every an onto function is a one-to-one function.

Part-II Chose the correct answer

1. Given a relation $R = \{(x, y) : y$ is greater than $x\}$. Which of the following ordered pairs belong to the relation R?

    A. $R = \{(2, 1), (-0.2, -0.21), (7, 7), (-2, -3), (0, -5)\}$.

    B. $R = \{(-4, -3), (-2, 0), (2,\ 3), (0, 5), (6, 9)\}$

C. $R = \{(2,1), (-4,3), (0.2,0.21), (-0.2,-0.21), (-2,-3)\}$

D. $R = \{(-2,0), (0.2,0.21), (-0.2,-0.21), (7,7), (-2,-3), (0,5)\}$

2. Let $f(x) = 1 - 3x$ and $g(x) = x^2 + 4x + 4$, then which one of the following statement is **not** correct?

   A. The domain of $f.g$ is the set of all real numbers.

   B. The domain of $f + g$ is the set of all real numbers.

   C. The domain of $\frac{f}{g}$ is the set of real numbers except -2.

   D. The range of $f + g$ is the set of all real numbers.

3. The figure shown below is the graphical representation of a relation. Which one of the following relations is represented by the shaded region?

   A. $\left\{(x,y): y \leq \frac{x}{2} \text{ and } |x| > 2\right\}$

   B. $\left\{(x,y): y \leq \frac{x}{2} \text{ and } |x| \geq 2\right\}$

   C. $\left\{(x,y): y \leq \frac{x}{2} \text{ and } x > 2\right\}$

   D. $\left\{(x,y): y \leq \frac{x}{2} \text{ and } x < 2\right\}$
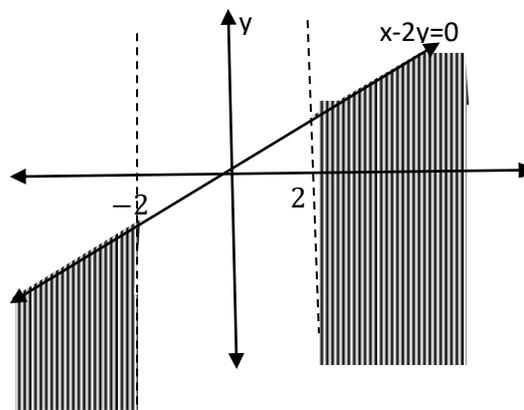
4. Let a function g be given as $g(x) = (1 + a)x - b - 2$ where a and b are positive real numbers. Which of the following is true about the graph of g?

   A.   The y- intercept is -2.

   B.   It is increasing for $a < 1$.

   C.   The x-intercept is $\frac{b+2}{a+1}$.

   D.   Its slope is a.

5. Which of the following functions is a bijective function?

   A. $f(x) = |x|$ **C.** $f(x) = x^2$

   B. $f(x) = 2x - 5$ **D.** All

Part-III Fill in the blank space

1. A function is called _____function if each element in the co-domain has at least one pre-image in the domain.

2. If $f(x) = 3x - 3$ and $g(x) = 2 - 3x$, then $(2f + 3g)(x)$ is equal to _____

3. Let $R = \{(x,\ y): y \text{ is shorter than } x\}$

   a. Does (y, y) belong to the relation? Why?

   b. Is true that if (x, y) belongs to R, then (y, x) belongs to R?

    c. If (x, y) and (y, z) belong to R, then is it true that (x, z) belongs to R?

4. If a function p is given by $p(x) = (1 - x)^2 + 4$, then what is the range of p?

5. Let $f = \{(-2,1), (0, 0), (4, -2), (5, -3)\}$, then the domain of F is_____

    and the range of F is _____

Part-IV Work out

1. Find the domain and range of each of the following relations:

    a. $R = \{(x, y): y = 3x - 4\}$.

    b. R is the relation which contains the set of order pairs $(x, y): x$ contains

        $-3, -1, 0, 1, 2, 4$ and $y = 2 - x$.

    c. $R = \{(x, y): y = \sqrt{4 - x^2}\}$.

    d. $R = \{(x, y): y \text{ is a sister of } x.\}$.

2. Sketch the graph of each of the following relations and find the domain and range.

    a. R is the relation which contains the set of order pairs $(x, y): y = x^3 - 1$.

    b. R is the relation which contains the set of order pairs $(x, y): y \leq x - 2$.

    c. R is the relation which contains the set of order pairs $(x, y): y > 3x - 1$.

    d. R is the relation which contains the set of order pairs $(x, y): y \geq 2x - 3$.

    e. R is the relation which contains the set of order pairs $(x, y): y \geq 2x$ and $y < x + 1$.

    f. R is the relation which contains the set of order pairs $(x, y): y < -x + 2$ and $y > x - 3$.

3. For the following graph (Figure 1.40), specify the relation and write down the domain and range:
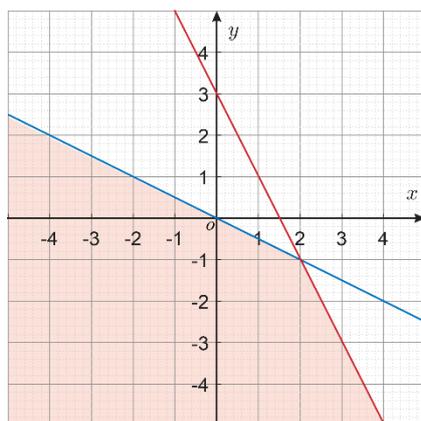
Figure 1.40

4. Let $f(x) = 2x^2 + 4x - 2$ and $g(x) = -x^2 - x + 7$

    a. Find: i) $f + g$ ii) $f - g$ iii) $(f + g)(-1)$ iv) $(2f - g)(1)$

    b. Determine the domain of $f - g$.

5. Let $f(x) = 2x^2 + 1$ and $g(x) = x - 3$, then

    a. Evaluate: i) fg ii) $\frac{f}{g}$ iii) $(fg)(1)$ iv) $\left(\frac{f}{g}\right)(2)$

    b. Find the domain of $\frac{f}{g}$.

6. Construct tables of values and sketch each of the following:

   a. $f(x) = -3x + 4$ **b.** $x - 3y + 2 = 1$

   c. $f(x) = 7 - 6x - x^2$

7. A mobile phone technician uses the linear function $C(t) = 3t + 200$ to determine the cost of repair where the time in the hours is t and $C(t)$ is the cost in Birr. How much will you pay if it takes him 3 hours to repair your mobile?

UNIT 2

Polynomial Functions

❈ Introduction

Dear learner, in unit one of this text book, you saw functions of the form $y = b, y = ax + b$ and $ax^2 + bx + c$. You also attempted to sketch their graphs. These functions are parts of a large class of functions called polynomial functions.

This unit is divided into five sections. In the first section we shall define polynomial function and discuss about degree, leading coefficient and constant term of this function.

In the second section we will discuss about the four fundamental operations on polynomial functions. These are addition, subtraction, multiplication and division of polynomial functions.

The third section is about theorems on polynomial functions. In this section we will see facts on polynomial functions. These are polynomial division theorem, remainder theorem and factor theorem.

In the fourth section we will discuss about the zeros of a polynomial function, the location theorem and the rational zero test.

The fifth section is about the properties of the graph of polynomial function. In this section we will see how the leading coefficient and the degree of a polynomial function determine the end property of the graph of the function.

In the six section we will see some applications of polynomial functions.

🎯 Learning Outcomes

At the end of this unit, you will be able to:

* ❋ define polynomial functions.
* ❋ perform the four fundamental operations on polynomials.
* ❋ apply theorems on polynomial functions to solve related problems.

❋ determine the rational and irrational zeros of polynomials.

❋ sketch the graphs of polynomial functions.

Unit contents

Section 1: Definition of polynomial function

Section 2: Operations on polynomial functions

Section 3: Theorem on polynomial functions

Section 4: Zeros of a Polynomial Function

Section 5: Graphs of Polynomial Functions

Section 6: Applications

Unit Learning Strategies

We would like to mention the following for your study of this chapter.

- Create a schedule and follow it. Scheduling makes you more practical.

- Dear learner, if you want to learn mathematics you have to do mathematics everywhere and every time. So, read repeatedly and do the worked examples until you get the steps and apply it to self-test exercises. Moreover, from your work you will find a pattern that you need to repeat it to solve the next review exercise

- When you attain what you had intended for a specific learning session, you should reward yourself with anything positive. This makes you more inspired to keep moving and cherish your learning and achievements.

Section 1 │ Definition of polynomial function

❇ Section Overview

Dear learner, you are familiar with functions like constant functions, linear functions and quadratic functions in unit one. These functions are parts of a large class of function called polynomial functions that we are going to define now. In addition, terms like degree, leading coefficient and constant term of a polynomial function will be defined.

Section Objectives

By the end of this section, you will be able to: -

- define polynomial functions.
- identify the degree, leading coefficient and constant term of a polynomial function.

? Do you remember the general form of a constant function, a linear function and a quadratic function? Can you write the domains and the ranges of these functions?

Activity 2.1

Classify the following functions as constant function, linear function, quadratic function or none of these:

a. $f(x) = 3x + 5$ **b.** $f(x) = x^4 - 3x + 4$ **c.** $f(x) = 6 + \frac{9}{5}x + 4x^2$

d. $k(x) = 6$ **e.** $g(x) = 7 - 5x$ **f.** $l(x) = 2\sqrt{x} + \sqrt{2}$

g. $h(x) = (x - 1)(x + 1)$ **h.** $h(x) = 6 + |x|$ **i.** $f(x) = 3x^3 - 2x^2 - 4x + 7$

Have you done it? Well!

A function f is a constant function if it can be written in the form $f(x) = a$, where a is a real number. The domain of f is the set of all real numbers and the range is the set containing only the number b.

A function f is a linear function if it can be written in the form $f(x) = ax + b$, where a and b are real numbers and $a \neq 0$. The domain of f is the set of all real numbers and the range is also the set of all real numbers.

A function f is a quadratic function if it can be written in the form $f(x) = ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$. The domain of f is the set of all real numbers and the range is not the set of all real numbers and it depends on the values of a, b and c.

Now, we give the definition of the class of function called polynomial function. You will see that constant function, linear function and quadratic function are special cases of this function.

Definition 2.1

Let n be non-negative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function

$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ is called a polynomial function in one variable x of degree n.

In the above definition of polynomial function

    i. $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are called **coefficients**.

    ii.   The number a_n is called the **leading coefficient** and $a_n x^n$ is the **leading term**.

    iii.   The number n (the exponent of the highest power of x) is called the **degree**.

    iv.   The number a_0 is called the constant term.

? Are constant, linear and quadratic functions polynomial functions?

What is the degree of a constant, a linear and a quadratic function?

$p(x) = a_0, \ a_0 \neq 0$ is a constant function, $p(x) = a_1 x + a_0, a_1 \neq 0$ is a linear function and $p(x) = a_2 x^2 + a_1 x + a_0, a_2 \neq 0$ is a quadratic function.

☞
- The domain of a polynomial function is the set of all real numbers.
- Constant, linear and quadratic functions are polynomial functions with degree 0, 1 and 2, respectively.
- The constant function $f(x) = 0$ is called the zero polynomial with no degree assigned to it.

Example 1

Find the degree, leading coefficient and constant term of the following polynomial functions.

a. $f(x) = 9$

b. $f(x) = -2x + 5$

c. $f(x) = 3x^2 + x - 3$

d. $f(x) = 3x^3 - 9x^2 + 5x + \frac{3}{2}$

e. $h(x) = -5x^4 + 8x^3 + 2x^2 - 3x + 7$

f. $g(x) = -2 + 3x^3 + \frac{2}{5}x^2 - x^4 + 4x^5 + 5x$

g. $g(x) = 2(x^4 + \frac{1}{2}x^2 - 2) + x + x^4 + 1$

Solution:

a. $f(x) = 9 = 9x^0$, it is a polynomial function with degree 0, leading coefficient 9 and constant term 9.

b. It is a polynomial function with degree 1, leading coefficient -2 and constant term 5.

c. It is a polynomial function with degree 2, leading coefficient 3 and constant term -3.

d. It is a polynomial function with degree 3, leading coefficient 3 and constant term $\frac{3}{2}$.

e. It is a polynomial function with degree 4, leading coefficient -5 and constant term 7.

f. You can rearrange the polynomial function g using commutative property of addition like $g(x) = 4x^5 - x^4 + 3x^3 + \frac{2}{5}x^2 + 5x - 2$ and it is a polynomial function with degree 5, leading coefficient 4 and constant term -2.

g. $g(x) = 2(x^4 + \frac{1}{2}x^2 - 2) + x + x^4 + 1 = 2x^4 + x^2 - 4 + x + x^4 + 1$

$$= 3x^4 + x^2 + x - 3.$$

Therefore, the degree is 4, the leading coefficient is 3 and the constant term is -3.

There are two ways to check whether a given function is not a polynomial function.

- Its domain is not all real numbers.
- It does not take the form or unable to transform it to the form
  $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where n is non negative-integer (0 or positive integers) and $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are real numbers with $a_n \neq 0$.

Example 2

Determine if the following are polynomial function or not.

1. $g(x) = 4x^{-2} + 3x^{-1} - 7$
2. $h(x) = 2^x$

Solution:

1. $g(x) = 4x^{-2} + 3x^{-1} - 7$ is not a polynomial function because -2 and -1 are not positive integers.

2. $h(x) = 2^x$ is not a polynomial function because it can't take the form of the general polynomial function.

Example 3

Which of the following are polynomial functions? For those which are polynomials, find the degree, leading coefficient and constant term.

a. $f(x) = 2x^2 - 9x^3 + \frac{2}{5}x$

b. $g(x) = (2 - x)(2 + x)^`$

c. $h(x) = 5x^{-4} + 6x^2 + 3$

d. $k(y) = \sqrt{y}$

e. $f(x) = 2x^{\frac{1}{3}} + 5x^{\frac{1}{2}} - x + 5$

f. $f(x) = \sqrt{2}x^3 + 5x^2 - \sqrt{\frac{2}{7}}$

Solution:

a. You can rearrange it as $f(x) = -9x^3 + 2x^2 + \frac{2}{5}x$ and it is a polynomial function with degree 3, leading coefficient -9 and constant term 0.

b. $g(x) = (2 - x)(2 + x) = 4 - x^2$, it is a polynomial function with degree 2, leading coefficient -1 and constant term 4.

c. $h(x) = 5x^{-4} + 6x^2 + 3$ is not a polynomial function because in the term $5x^{-4}$, $n = -4$ is not a positive integer.

d. $k(y) = \sqrt{y}$ is not a polynomial function because it cannot be written in the general form of polynomial function, because the variable y is inside a radical sign.

e. $f(x) = 2x^{\frac{1}{3}} + 5x^{\frac{1}{2}} - x + 5$ is not a polynomial function because $\frac{1}{3}$ and $\frac{1}{2}$ are not integers.

f. It is a polynomial function with degree 3, leading coefficient $\sqrt{2}$ and constant term $-\sqrt{\frac{2}{7}}$

☑ Checklist

Dear learner, now it is time to check your understanding about polynomial function. Read each question and put a mark (" √ ") in the box that is the appropriate choice to you.

	Yes	No
• Can you define a polynomial function?	☐	☐
• Can you identify the degree, leading coefficient and constant terms of a polynomial function?	☐	☐
• Can you give two examples of polynomial functions whose degree is 5?	☐	☐
• Can you give two examples of a function which are not polynomial functions?	☐	☐

If your response to the above check list questions is positive, proceed to do the self-test exercise given below and check your answer at the end of this module. If not please review the section again and again until you think you have clear understanding about it.

⚒ Self-Test Exercise 1

1. Find the degree, leading coefficient and constant term of the following polynomial functions.

a. $f(x) = -x^2 + 2x + 10$ **b.** $f(x) = 10x^3 + 5x^2 - 3x + 2$

c. $h(x) = -5x^5 + 4x^3 - 12$ **d.** $g(x) = -x^3 - 4x^2 + 3x^4 - 5 + 2x$

e. $h(x) = x^2 - 3x^4 + 2\left(5x^5 - 2x^2 + 4x^4 - \frac{5}{2}\right)$

2. Which of the following are polynomial functions? For those which are polynomials, find the degree, leading coefficient and constant term.

a. $f(x) = 2x^4 + 3x^3 - x - 6$ **b.** $k(x) = 7 - x + \frac{3}{8}x^5 - 8x^4$

c. $g(x) = 8x^3 - 17x^{-2} + x^{-1} - 2$ **d.** $h(t) = \sqrt{6}t + \sqrt{2}$

e. $h(y) = -\frac{5}{3}y^2 + \sqrt{2}y + \frac{1}{2}$ **f.** $f(x) = \frac{1}{x^3} + \frac{1}{x^4} + 1$

g. $k(x) = 3x^7 - 5x^2 + 2 - \dfrac{1 - x^2 + 4x^3 + x^7}{3}$ **h.** $f(x) = (x - 2)(x^2 + x - 1)$

i. $f(x) = \sqrt{(x^2 + 5)^4}$ **j.** $f(x) = |3x^2 + 8|$

Section 2 | Operations on Polynomial Functions

❈ Section Overview

Dear learner, remember that in algebra we can combine two real numbers using the operations addition, subtraction, multiplication and division to find another real number. You know the restriction we have to make when we use the operation division. Here, we will combine two or more polynomial functions using the operations addition, subtraction, multiplication and division and discuss the results obtained by the combination. To combine polynomial functions the knowledge of commutative, associative and distributive laws, like and unlike terms is very important.

Section objectives

By the end of this section, you will be able to:

- find the sum of two polynomial functions.
- compute the difference of two polynomial functions.
- determine the product of two polynomial functions.
- perform the quotient of two polynomial functions.

Activity 2.2

1. Which of the following pairs contain like terms?

    a. $2x$ and $5x$ **b.** $5a^2$ and $6a^2$ **c.** x^2 and $2x^3$

    d. $3x^2$ and $2\,y^2$ **e.** 3 and y **f.** x^5 and $6x^5$

2. For any three real numbers a, b and c, determine whether each of the following statements is true or false. Give reason for your answer.

    a. $a + b = b + a$ **b.** $a - b = b - a$ **c.** $(a + b) + c = a + (b + c)$

    d. $(ab)c = a(bc)$ **e.** $a(b + c) = ab + ac$ **f.** $a - (b + c) = a - b + c$

    g. $a - (b - c) = a - b + c$ **h.** $(a - b) + c = a + (-b + c)$

Have you done it? Well!

2.1 Addition of Polynomial Functions

Dear learner, you know that we can add two or more numbers using the operation addition (+) to get another number. For example, $2 + 3 = 5$, $10 + 8 + 9 = 27$, $-9 + 4 = -5$, etc. likewise, you can add two polynomial functions to get another polynomial function.

> ## Definition 2.1
>
> ☞ The sum of two polynomial functions f and g is written as $f + g$ and is defined as:
>
> $$(f + g)(x) = f(x) + g(x) \text{ for all real numbers } x.$$

In activity 2.2, you know the meaning of like terms. For example, $2x^2$ and $7x^2$ are like terms and we can group them as $2x^2 + 7x^2$ and add them to get $9x^2$. But $5x^2$ and $9x^3$ are not like terms. You can simply put $5x^2 + 9x^2$ for their sum.

> ☞ The sum of two polynomial functions is found by grouping like terms and adding the coefficients of the like terms.

Example 1

In each of the following, find the sum of $f(x)$ and $g(x)$.

a. $f(x) = 3x^3 - 6x^2 + 7x + 2$ and $g(x) = -2x^4 + 4x^3 + 8x^2 - 11x + 10$

b. $f(x) = 2x^4 - 5x^3 + x^2 + 9x + 4$ and $g(x) = x^4 + 3x^3 + 5x^2 - 5x + 9$

c. $f(x) = 2x^4 + 6x^3 + 8x^2 + x + 7$ and $g(x) = -2x^4 - 4x^3 + 7x^2 - 5x + 1$

d. $f(x) = \frac{1}{2}x^2 + \frac{4}{3}x - 5$ and $g(x) = \frac{3}{2}x^2 - \frac{2}{3}x + 3$

Solution:

a. $f(x) + g(x)$

$= (3x^3 - 6x^2 + 7x + 2) + (-2x^4 + 4x^3 + 8x^2 - 11x + 10)$

$= -2x^4 + (3x^3 + 4x^3) + (-6x^2 + 8x^2) + (7x - 11x) + (10 + 2)$ (Grouping like term)

$= -2x^4 + 7x^3 + 2x^2 - 4x + 12$ (Adding like term)

> **?** Which function has higher degree f or g?
>
> What is the degree of $f + g$?
>
> What relation do you see about the degrees of $f + g$ and g?

b. $f(x) + g(x)$

$= (2x^4 - 5x^3 + x^2 + 9x + 4) + (x^4 + 3x^3 + 5x^2 - 5x + 9)$

$= (2x^4 + x^4) + (-5x^3 + 3x^3) + (x^2 + 5x^2) + (9x - 5x) + (4 + 9)$ (Grouping like terms)

$= 3x^4 - 2x^3 + 6x^2 + 4x + 13$ (Adding like terms)

> **?** What is the degree of $f + g$?
>
> What relation do you observe about the degrees of f, g and $f + g$?

c. $f(x) + g(x)$

$= (2x^4 + 6x^3 + 8x^2 + x + 7) + (-2x^4 - 4x^3 + 7x^2 - 5x + 1)$

$= (2x^4 - 2x^4) + (6x^3 - 4x^3) + (8x^2 + 7x^2) + (x - 5x) + (7 + 1)$ (Grouping like terms)

$= 2x^3 + 15x^2 - 4x + 8$ (Adding like terms)

> **(?)** What do you see about the leading coefficients of f and g?
>
> What is the degree of $f + g$?
>
> Is the degree of $f + g$ lower than the degree of f and the degree g?

d. $f(x) + g(x) = \left(\frac{1}{2}x^2 + \frac{4}{3}x - 5\right) + \left(\frac{3}{2}x^2 - \frac{2}{3}x + 3\right)$

$\qquad\qquad\quad = \left(\frac{1}{2}x^2 + \frac{3}{2}x^2\right) + \left(\frac{4}{3}x - \frac{2}{3}x\right) + (-5 + 3)$ (Grouping like term)

$\qquad\qquad\quad = \frac{4}{2}x^2 + \frac{2}{3}x - 2$ (Adding like terms)

$\qquad\qquad\quad = 2x^2 + \frac{2}{3}x - 2$ (simplifying coefficient of $x^2, \frac{4}{2} = 2$)

Dear learner, from the above examples, we can take the following observations.

> ☞
> 1. If $f(x)$ and $g(x)$ have different degrees, the degree of $f(x) + g(x)$ is the same as the degree of $f(x)$ or the degree of $g(x)$ whichever has the highest degree.
> 2. If $f(x)$ and $g(x)$ have the same degree, the degree of the sum may be lower than or equal to the common degree.
> 3. The sum of two polynomial functions is a polynomial function.

2.2 Subtraction of Polynomial Functions

You know that we can find the difference of two numbers using the operation subtraction (-). For example, $9 - 5 = 4$, $21 - 13 = 9$ and $5 - 9 = -4$, etc. Similarly, we can find the difference of two polynomial functions. Grouping like terms is necessary to find the difference.

> **Definition 2.3**
>
> ☞ The difference of two polynomial functions f and g is written as $f - g$, and is defined as
>
> $$(f - g)(x) = f(x) - g(x) \text{ for all real numbers } x.$$

Example 2

In each of the following, find $f - g$.

a. $f(x) = 5x^3 + 6x^2 + 4x + 7$ and $g(x) = 4x^3 - 2x^2 + 5x - 2$.

b. $f(x) = -2x^3 + 5x^2 + 3x + 2$ and $g(x) = -2x^3 + 4x^2 + 8x - 7$.

c. $f(x) = 6x^5 + 5x^4 + 2x^3 - x^2 + 4x - 3$ and $g(x) = 5x^4 + x^3 + 4x^2 - 3x - 3$.

Solution:

a. $f(x) - g(x) = (5x^3 + 6x^2 + 4x + 7) - (4x^3 - 2x^2 + 5x - 2)$

$\qquad\qquad = 5x^3 + 6x^2 + 4x + 7 - 4x^3 + 2x^2 - 5x + 2$ (Removing brackets)

$\qquad\qquad = (5x^3 - 4x^3) + (6x^2 + 2x^2) + (4x - 5x) + (7 + 2)$ (Grouping like terms)

$\qquad\qquad = x^3 + 8x^2 - x + 9$ (Adding like terms)

> **(?)** What is the degree of $f - g$?
>
> What relation do you observe about the degrees of f, g and $f - g$?

b. $f(x) - g(x) = (-2x^3 + 5x^2 + 3x + 2) - (-2x^3 + 4x^2 + 8x - 7)$

$\qquad\qquad = -2x^3 + 5x^2 + 3x + 2 + 2x^3 - 4x^2 - 8x + 7$ (Removing brackets)

$\qquad\qquad = (-2x^3 + 2x^3) + (5x^2 - 4x^2) + (3x - 8x) + (2 + 7)$ (Grouping like terms)

$\qquad\qquad = 0x^3 + 1x^2 - 5x + 9$ (Adding like terms)

$\qquad\qquad = x^2 - 5x + 9.$

> **(?)** What do you see about the leading coefficients of f and g?
> What is the degree of $f - g$?
> Is the degree of $f - g$ lower than the degree of f and the degree g?

c. $f(x) - g(x) = (6x^5 + 5x^4 + 2x^3 - x^2 + 4x - 3) - (5x^4 + x^3 + 4x^2 - 3x - 3)$

$\qquad = 6x^5 + 5x^4 + 2x^3 - x^2 + 4x - 3 - 5x^4 - x^3 - 4x^2 + 3x + 3$ (Removing Brackets)

$\qquad = 6x^5 + (5x^4 - 5x^4) + (2x^3 - x^3) + (-x^2 - 4x^2) + (4x + 3x) + (-3 + 3)$

$\qquad$ (Grouping like terms)

$\qquad = 6x^5 + 0x^4 + 1x^3 - 5x^2 + 7x + 0$ (Adding like terms)

$\qquad = 6x^5 + x^3 - 5x^2 + 7x.$

> **(?)** Which function has higher degree f or g?
> What is the degree of $f - g$?
> What relation do you see about the degrees of $f - g$ and g?

Dear student, from the above examples, we can take the following observations.

> 1. If $f(x)$ and $g(x)$ have different degrees, the degree of $f(x) - g(x)$ is the same as the degree of $f(x)$ or the degree of $g(x)$ whichever has the highest degree.
> 2. If $f(x)$ and $g(x)$ have the same degree, the degree of the sum may be lower or equal to the common degree.
> 3. The difference of two polynomial functions is a polynomial function.

2.3 Multiplication of Polynomial Functions

Dear learner, you know how to find the product of two numbers using the operation multiplication ($\times$). For example, $2 \times 5 = 10$, $(-12) \times (-4) = 48$ and $-5 \times 9 = -45$, etc. You can also multiply one polynomial function by another polynomial function to get a polynomial function.

> ### Definition 2.4
>
> The product of two polynomial functions $f(x)$ and $g(x)$ is written as $f \cdot g$, and is defined as:
>
> $$(f \cdot g)(x) = f(x) \cdot g(x) \text{ for all real numbers } x.$$
>
> The product of two polynomials $f(x)$ and $g(x)$ is found by multiplying each term of one by every term of the other as shown in the following example.

Example 3

1. Find $f(x) \cdot g(x)$ where $f(x) = 2x + 3$ and $g(x) = 3x^2 - 5x + 6$.
2. Let $f(x) = x^2 - 2x + 2$ and $g(x) = 2x^3 - 4x^2 - 5x + 1$ then

   a. Find $f(x) \cdot g(x)$.

   b. Is $f(x) \cdot g(x)$ a polynomial function?

   c. Find the degree of f, g and $f \cdot g$.

   d. Is the degree of $f \cdot g$ equal to the sum of the degrees of f and g?

Solution:

1. $f(x) \cdot g(x) = (2x + 3)(3x^2 - 5x + 6)$

   $= 2x(3x^2 - 5x + 6) + 3(3x^2 - 5x + 6)$ (Distributive property)

   $= (2x)(3x^2) + (2x)(-5x) + (2x)(6) + (3)(3x^2) + (3)(-5x) + (3)(6)$

   (Distributive property)

   $= 6x^3 - 10x^2 + 12x + 9x^2 - 15x + 18$

   $= 6x^3 - 10x^2 + 9x^2 + 12x - 15x + 18$

   $= 6x^3 - x^2 - 3x + 18.$

2. a. $f(x) \cdot g(x) = (x^2 - 2x + 2)(2x^3 - 4x^2 - 5x + 1)$

   $= x^2(2x^3 - 4x^2 - 5x + 1) + (-2x)(2x^3 - 4x^2 - 5x + 1) + 2(2x^3 - 4x^2 - 5x + 1)$

   (Distributive property)

   $= 2x^5 - 4x^4 - 5x^3 + x^2 - 4x^4 + 8x^3 + 10x^2 - 2x + 4x^3 - 8x^2 - 10x + 2$

   (Distributive property)

   $= 2x^5 - 4x^4 - 4x^4 - 5x^3 + 8x^3 + 4x^3 + x^2 + 10x^2 - 8x^2 - 2x - 10x + 2$

   $= 2x^5 - 8x^4 + 7x^3 + 3x^2 - 12x + 2.$

b. Yes.

c. Degree of f is 2, degree of g is 3 and degree of $f \cdot g$ is 5.

d. Yes.

Dear student, from the above examples, you can have the following observations.

☞

1. For any two non-zero polynomial functions f and g, the degree of $f \cdot g$ is $m + n$ if the degree of f is m and the degree of g is n.

2. If either f or g is the zero polynomial then $f \cdot g$ becomes the zero polynomial and has no degree.

3. The product of two polynomial functions is a polynomial function.

Dear student, to find the product of two polynomial functions, we can also use a vertical arrangement for multiplication. This is shown by Example 2 of this section.

Example 4

Find $(x) \cdot g(x)$, if $f(x) = x^5 + 2x^4 - 3x^2 - x - 5$, $g(x) = 3x^2 - 4x + 2$.

Solution:

We arrange the polynomials in a vertical column and multiply each term of the second polynomial by each term of the first polynomial as indicated along with the following solution.

$$\left.\begin{array}{r} x^5 + 2x^4 - 3x^2 - x - 5 \\ 3x^2 - 4x + 2 \end{array}\right\}$$

$$2x^5 + 4x^4 \qquad - 6x^2 - 2x - 10 \text{ (Multiplying by 2)}$$
$$-4x^6 - 8x^5 \qquad + 12x^3 + 4x^2 + 20x \text{ (Multiplying by } -4x)$$
$$3x^7 + 6x^6 \qquad - 9x^4 - 3x^3 - 15x^2 \text{ (Multiplying by } 3x^2)$$

$$3x^7 + 2x^6 - 6x^5 - 5x^4 + 9x^3 - 17x^2 + 18x - 10 \text{ (Adding like terms vertically)}$$

Thus, $f(x). g(x) = 3x^7 + 2x^6 - 6x^5 - 5x^4 + 9x^3 - 17x^2 + 18x - 10$.

2.4 Division of Polynomial Functions

A number that takes the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$ is called a rational number. If b is positive integer, we can divide a by b to find two other integers q and r with $0 \leq r < b$ such that $\frac{a}{b} = q + \frac{r}{b}$. Here, a is called the **dividend**, b is called the **divisor**, q is called the **quotient** and r is called the **remainder**.

For example, to find q and r when 50 is divided by 3, you usually use a process called long division as follows:

Hence, $50 \div 3 = \frac{50}{3} = 16 + \frac{2}{3}$.

$$\begin{array}{r} \text{Quotient} \\ 16 \\ \text{Divisor} \longrightarrow 3)\overline{50} \longleftarrow \text{Dividend} \\ 3 \\ \overline{20} \\ 18 \\ \overline{2} \\ \text{Remainder} \end{array}$$

Here, 50 is the dividend, 3 is the divisor, 16 is the quotient and 2 is the remainder.

In almost a similar way, we can divide one polynomial by another polynomial

Activity 2.3

For each of the following, divide the number a by the number b to find two numbers q (quotient) and r (remainder) with $r < b$ such that $\frac{a}{b} = q + \frac{r}{b}$ if

   i. $a = 97$ and $b = 8$.   **ii.** $a = 168$ and $b = 5$.

   iii. $a = 287$ and $b = 15$.   **iv.** $a = 355$ and $b = 11$.

Definition 2.6

The division (Quotient) of two polynomial functions f and g is written as $f \div g$, and is defined as $(f \div g)(x) = f(x) \div g(x)$, for all real numbers x and $g(x) \neq 0$ (zero polynomial).

Dear learner, it is possible to divide one polynomial by another using a **long division process**. When you are asked to divide one polynomial function by another, stop the division process when you get a quotient and remainder that are polynomial functions and the degree of the remainder polynomial is less than the degree of the divisor polynomial.

Example 5

Find $(x) \div g(x)$, where $f(x) = 4x^3 + 4x^2 - x + 4$ and $g(x) = 2x - 1$.

Solution:

$$\text{Think } \frac{6x^2}{2x} = 3x$$
$$\text{Think } \frac{4x^3}{2x} = 2x^2 \qquad \text{Think } \frac{2x}{2x} = 1$$

$$
\begin{array}{r}
2x^2 + 3x + 1 \qquad \longleftarrow Quotient \\
2x - 1 \overline{)\ 4x^3 + 4x^2 - x + 4} \qquad \longleftarrow Dividend \\
\underline{4x^3 - 2x^2} \qquad \longleftarrow \text{Multiply } 2x-1 \text{ by } 2x^2 \\
6x^2 - x + 4 \qquad \longleftarrow \text{Result of subtraction} \\
\underline{6x^2 - 3x} \qquad \longleftarrow \text{Multiply } 2x-1 \text{ by } 3x \\
2x + 4 \qquad \longleftarrow \text{Result of subtraction} \\
\underline{2x - 1} \qquad \longleftarrow \text{Multiply } 2x-1 \text{ by } 1 \\
5 \qquad \longleftarrow \text{Result of subtraction}
\end{array}
$$

Divisor $\longrightarrow$ *Remainder* $\longrightarrow$

So, dividing $4x^3 + 4x^2 - x + 4$ by $2x - 1$ gives a quotient $2x^2 + 3x + 1$ and a remainder 5.

Checklist

Dear learner, now it is time to check your understanding about operations on polynomial functions.

Read each question and put a mark (" ✔ ") in the box that is the appropriate choice to you.

	Yes	No
• Can you find the sum of two polynomial functions?	☐	☐
• Can you determine the degree of the sum of two polynomial functions based on the degrees of the summands?	☐	☐
• Can you find the difference of two polynomial functions?	☐	☐
• Can you find the product of two polynomial functions?	☐	☐
• Can you determine the degree of the product of two polynomial functions based on the degrees of the polynomials?	☐	☐
• Can you find the quotient of two polynomial functions by using long division?	☐	☐

If your response to the above check list questions is positive, proceed to do the self-test exercise given below and check your answer at the end of this module. If not please review the section again so that you can have clear understanding

Self-Test Exercise 2

1. Find the sum of the polynomial functions $f(x)$ and $g(x)$.

    a. $f(x) = x^3 - 2x^2 - 3x + 4$ and $g(x) = -5x^4 + 3x^3 - 6x^2 + 2x + 3$.

    b. $f(x) = x^4 + 2x^3 - 5x^2 - 7x + 9$ and $g(x) = -5 + 7x + 8x^2 - 3x^3 - 2x^4$.

    c. $f(x) = 6x^5 + x^4 - 2x^3 + 6x^2 + x - 5$ and $g(x) = 4 + 2x - 3x^5 + 2x^4$.

    d. $f(x) = \sqrt{3}x^4 + 2x^3 - 5x^2 + x - 2\sqrt{5}$ and $g(x) = 2\sqrt{5} - 3x - x^2 + 2x^3 + 3\sqrt{3}x^4$.

2. In each of the following find $f(x) - g(x)$.

    a. $f(x) = -2x^3 + 5x^2 - 3x + 11$ and $g(x) = -9x^3 - x^2 + 6x - 3$.

    b. $f(x) = -\frac{1}{5}x^4 + 2x^3 + 3x^2 - \frac{3}{4}x + 2$ and $g(x) = \frac{2}{5}x^4 - x^3 - 2x^2 + \frac{7}{4}x + 3$

    c. $f(x) = -6x^5 + 2x^4 - 2x^3 + 2x^2 + 7x + 3$ and $g(x) = -2 + 5x - 6x^5 - 3x^4$.

    d. $f(x) = 3\sqrt{2}x^4 - 2x^3 + 5x^2 + x + 2\sqrt{7}$ and $g(x) = \sqrt{7} - 3x + x^2 - 2x^3 - 5\sqrt{2}x^4$.

3. Evaluate $f(x) \cdot g(x)$.

    a. $f(x) = 2x + 1$ and $g(x) = 4x^2 + x - 3$

    b. $f(x) = -x^2 + 3x - 1$ and $g(x) = 3x^2 - x + 2$

    c. $f(x) = -x^3 + 3x$ and $g(x) = x - 2x^2$

4. Find the product of $f(x)$ and $g(x)$ using vertical arrangement.

    a. $f(x) = -x^2 + 3x - 2$ and $g(x) = 4x + 3$.

    b. $f(x) = 2x^3 - x^2 - 5x + 1$ and $g(x) = 2x + 3x^2$.

    c. $f(x) = -x^3 + x^2 + 2x - 3$ and $g(x) = 2x^2 - 4x + 5$.

5. In each of the following, find the quotient $q(x)$ and the remainder $r(x)$ when $f(x)$ is divided by $g(x)$.

    a. $f(x) = 2x^3 + 3x^2 - x - 1$ and $g(x) = x - 1$.

    b. $f(x) = 6x^3 - 5x^2 + 9x + 2$ and $g(x) = -2x + 1$.

    c. $f(x) = -x^3 + 3x^2 + 8$ and $g(x) = x - 2$.

Section 3   Theorems on Polynomials

❊ Section Overview

Dear learner, in this section you will see three facts about polynomial functions. The facts are stated by theorems. The theorems are the polynomial division theorem, the remainder theorem and the factor theorem.

> **Section objectives**
>
> **By the end of this section, you will be able to: -**
>
> - state the polynomial division theorem.
> - apply the polynomial division theorem.
> - state and apply the factor theorem.

3.1 Polynomial Division Theorem

Activity 2.4

For each of the following, divide the number a by the number b to find two numbers q (quotient) and r (remainder) with $r < b$ such that $a = qb + r$ if

    i. $a = 88$ and $b = 5$.

    ii. $a = 305$ and $b = 6$.

    iii. $a = 354$ and $b = 17$.

    iv. $a = 444$ and $b = 111$.

Theorem 2.1 (Polynomial Division Theorem)

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

where, $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$.

If the remainder $r(x)$ is zero, then we say $d(x)$ divides $f(x)$ exactly.

Example 1

For each of the following pairs of polynomials, find polynomials $q(x)$ and $r(x)$ such that $f(x) = d(x)q(x) + r(x)$.

a. $f(x) = x^3 - 1$; $d(x) = x - 1$.

b. $f(x) = -2x^3 + x^2 - 3x + 7$; $d(x) = x^2 - 1$.

Solution:

a.

$$
\begin{array}{r}
x^2 + x + 1 \\
x - 1 \overline{)\, x^3 \qquad\quad - 1}
\end{array}
$$

$x^3 - x^2$	$(x^2)(x-1)$
$x^2 \quad -1$	$(x^3 - 1) - (x^3 - x^2)$
$x^2 - x$	$x(x-1)$
$x - 1$	$(x^2 - 1) - (x^2 - x)$
$x - 1$	$(1)(x-1)$
0	$(x-1) - (x-1)$

$q(x) = x^2 + x + 1$ and $r(x) = 0$. Since $r(x) = 0$ we say that $x^3 - 1$ is exactly divisible by $x - 1$ and $x^3 - 1 = (x^2 + x + 1)(x - 1) + 0 = (x^2 + x + 1)(x - 1)$.

? Can you check by expanding $(x^2 + x + 1)(x - 1)$,

that $(x^2 + x + 1)(x - 1) = x^3 - 1$?

b.

$$
\begin{array}{r}
-2x + 1 \\
x^2 - 1 \overline{)\, -2x^3 + x^2 - 3x + 7}
\end{array}
$$

$$
\begin{array}{r}
-2x^3 \qquad +2x \\
\hline
x^2 - 5x + 7 \\
x^2 - 1 \\
\hline
-5x + 8
\end{array}
$$

The quotient is $q(x) = -2x + 1$ and the remainder is $r(x) = -5x + 8$ such that
$$-2x^3 + x^2 - 3x + 7 = (-2x + 1)(x^2 - 1) + (-5x + 8).$$

(?) Can you check $-2x^3 + x^2 - 3x + 7 = (-2x + 1)(x^2 - 1) + (-5x + 8)$?

3.2 Remainder Theorem

Activity 2.5

Find the remainder when the polynomial $f(x)$ is divided by the polynomial $x - c$ for the given number c. Compare the result you obtained with $f(c)$.

    a. $f(x) = x^2 - x + 3;\ c = -2$

    b. $f(x) = x^3 + 2x^2 - x - 5\ ;\ c = 1$

Theorem 2.2 (Remainder theorem)

Let $f(x)$ be a polynomial of degree greater than or equal to 1 and let c be any real number. If $f(x)$ is divided by the linear polynomial $(x - c)$, then the remainder is $f(c)$.

Example 2

In each of the following pairs of polynomials, use remainder theorem to find the remainder when $f(x)$ is divided by $d(x)$.

    a. $f(x) = 2x^3 + 5x^2 + 3x + 2;\ \ d(x) = x + 1.$

    b. $f(x) = x^4 + 3\ ; d(x) = x - 2.$

    c. $f(x) = x^{35} + 5x^{24} - x^9 + 9;\ d(x) = x - 1.$

    d. $f(x) = 55x^{201} + 100;\ d(x) = x + 1.$

Solution:

    a. $d(x) = x + 1 = x - (-1)$. Therefore $c = -1$ and the remainder is

        $f(c) = f(-1) = 2(-1)^3 + 5(-1)^2 + 3(-1) + 2 = -2 + 5 - 3 + 2 = 2.$

(?) Can you check that the remainder is 2 by dividing $f(x) = 2x^3 + 5x^2 + 3x + 2$ by $d(x) = x + 1$ using long division?

    b. $d(x) = x - 2$, therefore $c = 2$ and the remainder is

        $f(c) = f(2) = 2^4 + 3 = 16 + 3 = 19.$

(?) Can you check that the remainder is 19 by dividing $f(x) = x^4 + 3$ by $d(x) = x - 2$ using long division?

c. $d(x) = x - 1$, therefore $c = 1$ and the remainder is

$$f(c) = f(1) = 1^{35} + 5(1)^{24} - 1^9 + 9 = 14.$$

d. $d(x) = x + 1 = x - (-1)$, therefore $c = -1$ and the remainder is

$$f(-1) = 55(-1)^{201} + 100 = 45.$$

Example 3

a. Find the value of b such that when $3x^3 - 4x^2 + b - 5$ is divided by $x - 2$, the remainder is 10.

b. Find the value of a and b such that when $x^3 + ax^2 + bx - 9$ is divided by $x + 1$ and $x - 2$ the remainder is 4 and -5, respectively.

Solution:

a. Let $f(x) = 3x^3 - 4x^2 + b - 5$, by the remainder theorem when $f(x)$ is divided by $x - 2$ the remainder is $f(2) = 3(2)^3 - 4(2)^2 + b - 5 = b + 3$. Since the remainder is given as 10, we have $b + 3 = 10$ and solving for b, we have $b = 7$.

b. Let $f(x) = x^3 + ax^2 + bx - 9$. When $f(x)$ is divided by $x + 1$ the remainder is

$$f(-1) = (-1)^3 + a(-1)^2 + b(-1) - 9 = a - b - 10.$$

Since the remainder is 4, $a - b - 10 = 4$,

$$a - b = 14 \ . \ . \ . \ (1).$$

When $f(x)$ is divided by $x - 2$ the remainder is

$$f(2) = (2)^3 + a(2)^2 + b(2) - 9 = 4a + 2b - 1.$$

Since the remainder is -5, $4a + 2b - 1 = -5$,

$$4a + 2b = -4$$

$$2a + b = -2 \ . \ . \ . \ (2).$$

From (1) and (2) we have $a = 4$ and $b = -10$.

3.3 Factor Theorem

Remember that in the case of multiplication of polynomials, we multiply two or more polynomials to find another polynomial.

For example, $(x + 1)(2x - 1) = 2x^2 + x - 1$.

The polynomial $2x^2 + x - 1$ is called product or multiple and $(x + 1)$ and $(2x - 1)$ are called **factors**. Factoring a polynomial means writing it as the product of its polynomial factors.

The following theorem is known as the **factor theorem**. It is very helpful to check whether a linear polynomial is a factor of a given polynomial or not.

Activity 2.6

Let $f(x) = x^3 + 4x^2 + x - 6$.

a. Find $f(1)$.

b. Find the quotient $q(x)$ and the remainder $r(x)$ when $f(x)$ is divided by $x - 1$.

c. Express $f(x)$ as $f(x) = (x - 1)q(x) + r(x)$.

d. Is $(x - 1)$ a factor of $f(x)$?

Theorem 2. 3 (Factor theorem)

Let $f(x)$ be a polynomial of degree greater than or equal to one, and let c be any real number, then

1. if $f(c) = 0$ then $x - c$ is a factor of $f(x)$.

2. if $x - c$ is a factor of $f(x)$ then $f(c) = 0$.

Example 4

a. Show that $x + 2$ is a factor of $f(x) = x^2 + 5x + 6$.

b. Show that $x + 1$ and $x - 2$ are factors but $x + 2$ is not factor of

$f(x) = x^4 - x^3 - x^2 - x - 2$.

Solution:

a. Since $x + 2 = x - (-2)$ has the form $x - c$, the value of $c = -2$.

Now, $f(c) = f(-2) = (-2)^2 + 5(-2) + 6 = 0$, then by the factor theorem $x + 2$ is a

factor of $x^2 + 5x + 6$.

b. Since $x + 1 = x - (-1)$ has the form $x - c$, the value of $c = -1$.

$f(c) = f(-1) = (-1)^4 - (-1)^3 - (-1)^2 - (-1) - 2 = 0$. Then by the factor theorem

$x + 1$ is a factor of $f(x)$.

Since $x - 2$ has the form $x - c$, the value of $c = 2$.

$f(c) = f(2) = (2)^4 - (2)^3 - (2)^2 - (2) - 2 = 0$. Then by the factor theorem $x - 2$ is a

factor of $f(x)$.

Since $x + 2 = x - (-2)$ has the form $x - c$, the value of $c = -2$.

$f(c) = f(-2) = (-2)^4 - (-2)^3 - (-2)^2 - (-2) - 2 = 20 \neq 0$. Then by the factor

theorem $x + 2$ is not a factor of $f(x)$.

Example 5

a. Find the number k such that $x + 1$ is a factor of $kx^3 + 2x^2 - 3kx + 2$.

b. Find the values of a and b in the polynomial $x^3 + ax^2 + bx + 6$ such that $x + 1$ and $x - 2$

are its factors.

Solution:

a. Let $f(x) = kx^3 + 2x^2 - 3kx + 2$.

$x + 1 = x - (-1)$ is a factor of f implies $f(-1) = 0$.

That is, $f(-1) = 0$

$$k(-1)^3 + 2(-1)^2 - 3k(-1) + 2 = 0$$

$$-k + 2 + 3k + 2 = 0$$

$$2k + 4 = 0$$

$$k = -2$$

b. Let $f(x) = x^3 + ax^2 + bx + 6$.

$x + 1$ is a factor of $f(x)$ implies $f(-1) = 0$.

On the other hand,

$f(-1) = (-1)^3 + a(-1)^2 + b(-1) + 6 = a - b + 5$.

Therefore, $a - b + 5 = 0$

$$a - b = -5 \ . \ . \ . \ (1)$$

$x - 2$ is a factor of $f(x)$ implies $f(2) = 0$.

On the other hand,

$f(2) = (2)^3 + a(2)^2 + b(2) + 6 = 4a + 2b + 14$.

Therefore, $4a + 2b = -14$

$$2a + b = -7 \ . \ . \ . \ (2)$$

From (1) and (2) we have $a = -4$ and $b = 1$.

Checklist

Dear learner, now it is time to check your understanding about the theorems on polynomial functions. Read each question and put a mark (" ✓ ") in the box that is the appropriate choice to you.

	Yes	No
• Can you state the polynomial Division Theorem?	☐	☐
• Can you divide a polynomial $f(x)$ by a non-zero polynomial $d(x)$ and find a quotient $q(x)$ and a remainder $r(x)$ that satisfy the polynomial division theorem.	☐	☐
• Can you state the Remainder Theorem?	☐	☐
• Can you find the remainder when a polynomial $p(x)$ is divided by $x - a$?	☐	☐
• Can you state the Factor Theorem?	☐	☐

- Can you determine whether $x - c$ is a factor of a polynomial ☐ ☐

  $f(x)$ or not?

If your response to the above check list questions is positive, proceed to do the self-test exercise given below and check your answer at the end of this module. If not please review the section again and again until you think you have clear understanding about it.

✖ Self-Test Exercise 3

1. For each of the following pairs of polynomials, find the quotient $q(x)$ and the remainder

$r(x)$ that satisfy the polynomial division theorem if

a. $f(x) = 5x^2 + x - 2; \ d(x) = -x + 1.$

b. $f(x) = 2x^3 + 3x^2 - 8x - 4; \ d(x) = x^2 - 2x + 1.$

c. $f(x) = 3x^4 - 5x^3 + 12x - 4; \ d(x) = x^3 + 1$

d. $f(x) = -x^3 - 4x^2 + x + 2 ; \ d(x) = x + 2.$

e. $f(x) = x^4 ; \ d(x) = 3 - x.$

2. In each of the following, use the remainder theorem to find the remainder when $f(x)$ is

divided by $d(x)$.

a. $f(x) = 2x^3 - 3x^2 - 1, d(x) = x - 2$

b. $f(x) = -3x^3 - 2x^2 + x - 5, \ d(x) = x + 1.$

c. $f(x) = 2x^{19} - 4x^{12} + 9x - 62, \ d(x) = x - 1.$

d. $f(x) = x^{16} + 8x^3 + 99, \ d(x) = x + 1.$

e. $f(x) = 2x^3 + 3, \ d(x) = x - \frac{1}{2}.$

3. When $2x^4 - 2x^2 + bx - 4$ is divided by $x + 2$, the remainder is 5. Find the value of b.

4. Find the values of a and b such that when $5x^3 - ax^2 + bx - 1$ is divided by $x - 1$ and

$x + 1$ the remainder is -2 and 4, respectively.

5. Show that $x - 3$ is a factor of $f(x) = x^2 - 2x - 3$.

6. Which of the following is a factor of $f(x) = x^3 - 2x^2 - 5x + 6$?

a. $x - 1$ **b.** $x - 3$ **c.** $x + 1$ **d.** $x + 2$

7. In each of the following, determine whether $x - c$ is a factor of $f(x)$.

a. $f(x) = x^4 - x^3 - 13x^2 + x + 12; \ c = 1.$

b. $f(x) = 6x^3 - 5x^2 - 12x - 4; \ c = -\frac{1}{2}.$

c. $f(x) = x^3 - 3x^2 + 4x - 3; \ c = 2.$

8. In each of the following, find a number k satisfying the given condition.

a. $x - 1$ is a factor of $\ 2kx^3 + x^2 + 5x + 2.$

  b. $x + 2$ is a factor of $x^4 - 2kx^3 - 3kx^2 - x + 1$.

9. Find the values of a and b in the polynomial $x^4 + 2ax^3 - 3x^2 - 3bx + 8$ such that

  $x - 1$ and $x + 2$ are its factors.

Section 4   Zeros of a Polynomial Function

Dear student, from your grade 9 mathematics lesson, you know how to find the solution or root of linear and quadratic equations. That is the roots of equations of the form $ax + b = 0$ and

$ax^2 + ba + c = 0$.

In this section, we will define another name for a root of the equation $f(x) = 0$. We call this name the zero of $f(x)$. We will also try to find a zero of some polynomial functions.

In relation with a zero of a polynomial function, we will also discuss on some terms like multiplicity, location theorem and rational zeros.

Section Outcomes

By the end of this section, you will be able to: -

- compute the zeros of some polynomial functions.

- Verify that a polynomial function has a zero between two real numbers a and b with $a < b$.

- find the rational zeros of a polynomial function.

Activity 2.7

Find the solution of the following equations.

  a. $2x + 3 = -5 + 3x$ **b.** $\frac{2}{3}x + \frac{5}{4} = 2(x - \frac{1}{8})$

  c. $(2x - 3)(4 + 2x) = 0$ **d.** $x^2 - 5x + 6 = 0$

  e. $x^2 + 2x + 1 = 0$ **f.** $x^2 + 4 = 0$

4.1 Definition of Zeros of a Polynomial Function.

Definition 2.7

For a polynomial function $f(x)$ and a real number c, if $f(c) = 0$ then c is a **zero** of f.

1. If $x - c$ is a factor of $f(x)$, then c is the zero of $f(x)$.
2. If c is the zero of $f(x)$, then $x - c$ is a factor of $f(x)$.
3. If c is the zero of $f(x)$, then c is the root or solution of the equation $f(x) = 0$.

Example 1

a. Find the zeros of $f(x) = (x - 2)(3x + 4)(1 - 4x)$.

b. Determine the zeros of $f(x) = x^4 - 13x^2 + 36$.

c. If $x = 1$ is one zero of $f(x) = x^3 - 6x^2 + 11x - 6$, then find the rest of real zeros and rewrite $f(x)$ as a product of its factors.

Solution:

a. $f(x) = 0$

$(x - 2)(3x + 4)(1 - 4x) = 0$

$x - 2 = 0$ or $3x + 4 = 0$ or $1 - 4x = 0$

$x = 2$ or $x = -\dfrac{4}{3}$ or $x = \dfrac{1}{4}$

Therefore, $x = 2$, $x = -\dfrac{4}{3}$ and $x = \dfrac{1}{4}$ are the zeros of $f(x)$.

b. $f(x) = 0$

$x^4 - 13x^2 + 36 = 0$

$(x^2)^2 - 13x^2 + 36 = 0$ (Let $y = x^2$)

$y^2 - 13y + 36 = 0$

$(y - 4)(y - 9) = 0$

$(x^2 - 4)(x^2 - 9) = 0$

$(x - 2)(x + 2)(x - 3)(x + 3) = 0$

$x = 2$ or $x = -2$ or $x = 3$ or $x = -3$

Therefore, $x = 2$, $x = -2$, $x = 3$ and $x = -3$ are the zeros of $f(x)$.

c. Since $x = 1$ is a zero of $f(x)$, by the factor theorem $x - 1$ is one factor of $f(x)$.

? Can you show using long division, that is by dividing $x^3 - 6x^2 + 11x - 6$ by $x - 1$ the other factor is $x^2 - 5x + 6$?

Further factoring $x^2 - 5x + 6$ we have,

$x^2 - 5x + 6 = x^2 - 2x - 3x + 6 = x(x - 2) - 3(x - 2) = (x - 2)(x - 3)$.

Hence, $x^3 - 6x^2 + 11x - 6 = 0$ is the same as $(x - 1)(x - 2)(x - 3) = 0$.

Therefore, $x = 1$, $x = 2$ and $x = 3$ are the zeros of $f(x)$.

Dear student, from the beginning of section 4 up to this point you were looking how to find the zero of a polynomial function. Now, the meaning of multiplicity of a zero of a polynomial function $f(x)$ is going to be defined for you.

■ Example 2

Find a polynomial function $f(x)$ of degree two whose zeros are 1, -2 and satisfying the condition $f(3) = 30$.

Solution:

Let $f(x) = k(x - 1)(x + 2)$ for some number k. Clearly 1 and -2 are zeros of $f(x)$. To find k, since $f(3) = k(3 - 1)(3 + 2) = k(2)(5) = 10k = 30$, implies $k = 3$. Therefore, the polynomial function of degree two is $f(x) = 3(x - 1)(x + 2) = 3x^2 + 3x - 6$.

4.2 Location Theorem

Activity 2.8

In each of the following, determine whether the zeros of the polynomial function are rational, irrational, or neither.

a. $f(x) = x(x + 2)(x - \frac{3}{2})$.   **b.** $f(x) = (x + \sqrt{2})(x - \sqrt{3})$.   **c.** $f(x) = x^2 + 1$.

Consider $f(x) = x^2 - 3$. $-\sqrt{3}$ and $\sqrt{3}$ are the zeros of f and they are irrational numbers.

The table of values of the given functions for $-2 \leq x \leq 2$ and x is an integer, is the following.

x	-2	-1	0	1	2
$f(x)$	1	-2	-3	-2	1

Since $f(-2) = 1 > 0$ and $f(-1) = -2 < 0$, we see that the value of $f(x)$ changes sign from positive to negative between -2 and -1. And observe that one of the irrational roots $-\sqrt{3} \approx -1.73$ lies between these two numbers. We also see that the value of $f(x)$ changes sign from negative to positive between 1 and 2. Similarly observe that the second irrational root $\sqrt{3} \approx 3.73$ lies between these two numbers.

The following theorem which is called the location theorem helps to locate the real zeros of a polynomial function.

Theorem 2.4 Location Theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a)$ and $f(b)$ have opposite signs, then there is at least one zero of f between the numbers a and b.

Example 3

Let $f(x) = x^4 - 2x^3 - 4x^2 + 4x + 4$. Construct a table of values and use the location theorem to locate the zeros of f for the integers x and $-3 \le x \le 3$.

Solution:

x	-3	-2	-1	0	1	2	3
$f(x)$	91	12	-1	4	3	-4	7

Since $f(-2) = 12 > 0$ and $f(-1) = -1 < 0$, we see that the value of $f(x)$ changes sign from positive to negative between -2 and -1. Hence by the location theorem there is a zero of $f(x)$ between $x = -2$ and $x = -1$.

Since $f(-1) = -1 < 0$ and $f(0) = 4 > 0$, we see that the value of $f(x)$ changes sign from negative to positive between -1 and 0. Hence by the location theorem there is a zero of $f(x)$ between $x = -1$ and $x = 0$. Similarly, there are zeros of $f(x)$ between $x = 1$ and $x = 2$ and between $x = 2$ and $x = 3$.

Example 4

Using the location theorem, show that the polynomial function

$f(x) = x^6 - 2x^5 - 4x^2 + 4x + 4$ has a zero between $x = 1$ and $x = 2$.

Solution:

$f(1) = (1)^6 - 2(1)^5 - 4(1)^2 + 4(1) + 4 = 3 > 0$

$f(2) = (2)^6 - 2(2)^5 - 4(2)^2 + 4(2) + 4 = -4 < 0$

Hence, $f(1)$ is positive and $f(2)$ is negative and by the location theorem $f(x)$ has a zero between $x = 1$ and $x = 2$.

4.3 Rational Zero Test

Dear student, you know that a rational number is a number that takes the form $\frac{a}{b}$ where a and b are integers and $b \ne 0$.

We use Theorem 2.5 to find the possible rational zeros of a polynomial equation $(x) = 0$. If $\frac{a}{b}$ is among the possible rational roots and $f\left(\frac{a}{b}\right) = 0$ then we call $\frac{a}{b}$ a rational root of $f(x) = 0$.

> **Theorem 2.5 Rational Zero Test**
>
> Suppose that all the coefficients of the polynomial function described by
>
> $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ where $a_0, a_1, \ldots, a_n$ are integers with $a_n \ne 0$ and $a_0 \ne 0$. If $\frac{p}{q}$ in lowest term is a zero of $f(x)$, then p is a factor of a_0 and q is a factor of a_n.

Steps to find the rational zeros of a polynomial function $f(x)$.

1. Arrange the polynomial in descending order so that it takes the form
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0.$$

2. Write down all the factors of the constant term a_0. These are all the possible values of p.

3. Write down all the factors of the leading coefficient a_n. These are all the possible values of q.

4. Write down all the possible values of $\frac{p}{q}$. Remember that since factors can be negative, $\frac{p}{q}$ and $-\frac{p}{q}$ must both be included. Simplify each value and cross out any duplicates.

5. Identify those values of $\frac{p}{q}$ for which $f\left(\frac{p}{q}\right) = 0$. These are all the rational zeros of $f(x)$.

Note that for a polynomial function $f(x)$, the rational zeros of $f(x)$ and the rational roots of the equation $f(x) = 0$ are the same

Example 5

In each of the following, find all the rational zeros of the polynomial.

a. $f(x) = x^2 + x - 2$
b. $f(x) = x^3 - x + 1$
c. $f(x) = 6x^3 + 13x^2 + x - 2.$
d. $f(x) = \frac{1}{2}x^3 + x^2 - \frac{1}{2}x - 1$

Solution:

a. $f(x) = x^2 + x - 2$ has leading coefficient $a_2 = 1$ and constant term $a_0 = -2$.

Possible values of p are factors of -2. These are $\pm 1, \pm 2$.

Possible values of q are factors of 1. These are ± 1.

The possible rational zeros $\frac{p}{q}$ are $\pm 1, \pm 2$. Since $f(x)$ is a polynomial function of degree 2, it has at most 2 zeros, and from the four possible rational zeros at most 2 can be the zeros of f. We can check this using the table below

x	-2	-1	1	2
$f(x)$	0	-2	0	4

Therefore, the zeros of $f(x)$ are -2 and 1.

b. $f(x) = x^3 - x + 1$, the leading coefficient is 1 and the constant term is 1. Hence, the possible rational zeros are ± 1.

Check that $f(1) = f(-1) = 1 \neq 0$. So, we can conclude that the given polynomial has no rational zero. Use the location theorem to show that f has a zero between -2 and -1. That is, $f(x)$ has a real zero but it has no rational zero.

c. The leading coefficient is $a_3 = 6$ and the constant term is $a_0 = -2$.

Possible values of p are factors of -2. These are $\pm 1, \pm 2$.

Possible values of q are factors of 6. These are $\pm 1, \pm 2, \pm 3$ and ± 6.

The possible rational zeros $\frac{p}{q}$ are ±1, $\pm\frac{1}{2}$, $\pm\frac{1}{3}$, $\pm\frac{1}{6}$, ±2, $\pm\frac{2}{3}$. Since $f(x)$ is a polynomial function of degree 3 it has at most three zeros and from the 12 possible rational zeros at most 3 can be the zeros of f.

Check that $f\left(-\frac{1}{2}\right) = f(-2) = f\left(\frac{1}{3}\right) = 0$.

Using the factor theorem, we can factorize $f(x)$ as $f(x) = (3x - 1)(2x + 1)(x + 2)$ with $-\frac{1}{2}$, -2 and $\frac{1}{3}$ are the only rational zeros of f.

d. $f(x) = \frac{1}{2}x^3 + x^2 - \frac{1}{2}x - 1 = \frac{1}{2}(x^3 + 2x^2 - x - 2) = \frac{1}{2}k(x)$

where, $k(x) = x^3 + 2x^2 - x - 2$ has integer coefficients and the same zeros as $f(x)$.

$k(x)$ has a constant term -2 and leading coefficient 1. The possible rational zeros $k(x)$ are ±1 and ± 2. You can check that $k(1) = k(-1) = k(-2) = 0$

Therefore, the zeros of $f(x)$ are ±1 and -2.

Checklist

Dear learner, now it is time to check your understanding about this section. Read each question and put a mark (" ✓ ") in the box that is the appropriate choice to you.

	Yes	No
• Can you define the zeros of a polynomial function?	☐	☐
• Can we say that the zero of $f(x)$ and the root of the equation $f(x) = 0$ have the same meaning?	☐	☐
• Can you state the Location Theorem?	☐	☐
• Can you state the Rational Zero Test?	☐	☐
• Can you find the rational zeros of a polynomial function.	☐	☐
• Can we say that all the rational zeros of $f(x)$ and all the rational roots of $f(x) = 0$ are the same?		

If your response to the above check list questions is positive, proceed to do the self-test exercise given below and check your answer at the end of this module. If not please review the section again and again until you think you have clear understanding about it.

Self-Test Exercise 4

1. Find the zeros of the following functions.

    a. $f(x) = (x - 2)(x + 4)(5x - 2)$ **b.** $f(x) = x^4 - 10x^2 + 9$

    c. $f(x) = x^4 - x^2 - 6$ **d.** $f(x) = 2x^3 + 5x^2 - 3x$

2. Find the zeros of each of the following polynomial functions.

    a. $k(t) = (t + \frac{2}{3})^3 t^{10}$.
                  b. $g(x) = 5(x + \sqrt{2})^2 (x + 2)^3 (1 + 3x)$.

3. Find a polynomial $f(x)$ of degree two whose zeros are -1, 2 and satisfying the condition

    $f(3) = 12$.

4. Use the location theorem to verify that $f(x)$ has a zero between a and b.

    a. $f(x) = x^3 - x^2 - 2x + 1$; $a = 1$, $b = 2$.

    b. $f(x) = -2x^5 - x^4 + 2x^3 - 2x^2 + x + 1$; $a = -1, b = 0$.

5. For each of the following polynomials, find all possible rational zeros:

    a. $f(x) = 2x^2 + 7x + 3$
                 b. $f(x) = 6x^3 - 13x^2 + 4x + 3$

Section 5   Graphs of Polynomial Functions

❈ Section Overview

Dear learner, in unit 1, you discussed how to draw the graphs of the polynomial functions of degree zero, one and two. You saw that the graph of a linear function is a straight line and the graph of a quadratic function is a parabola. In this section you will learn about the properties of graphs of polynomial functions.

> **Section objectives**
>
> **By the end of this section, you will be able to: -**
>
> - sketch the graph of a quadratic function.
> - write the behaviors of the graph of odd degree polynomial functions.
> - explain the behaviors of the graph of even degree polynomial function.

Activity 2.9

Use table of values to sketch the graphs of $f(x) = 2x + 3$ and $g(x) = -2x + 3$. Include the x-intercept and y-intercept when you make the table of values. Sketch the graphs on the same xy-plane.

Example 1

For the function $f(x) = x^2 + 2x - 3$,

    a. Find the intercepts.

b. Using completing the square method, rewrite f as $f(x) = -4 + (x+1)^2$ and find the turning point.

c. Complete the table of values below.

d.

x	-4	-3	-2	-1	0	1	2
$y = f(x)$							

e. Sketch the graph of f, first by plotting the points (x, y) and then joining them by a smooth curve (**a smooth curve is a curve that has no sharp corner**).

f. Find the domain and range of f.

Solution:

a. $f(x) = y = x^2 + 2x - 3 = (x+3)(x-1)$. By making $f(x) = y = 0$ and solving the equation $x^2 + 2x - 3 = (x+3)(x-1) = 0$, we get $x = -3$ or $x = 1$. By making $x = 0$ we have $y = -3$. Thus, $x = -3$ and $x = 1$ are the x-intercepts and $y = -3$ is the y-intercept.

b. $f(x) = x^2 + 2x - 3 = (x^2 + 2x + 1) - 1 - 3 = (x+1)^2 - 4 = -4 + (x+1)^2$. Since $(x+1)^2 \geq 0$ for all real numbers x, $f(x) \geq -4$ for all values of x. Therefore, -4 is the minimum value of f. This minimum value of f is attained when $x = -1$. **The point $(-1, -4)$ is called turning point or vertex of the graph of f.**

c.

x	-4	-3	-2	-1	0	1	2
$y = f(x)$	5	0	-3	-4	-3	0	5

d.
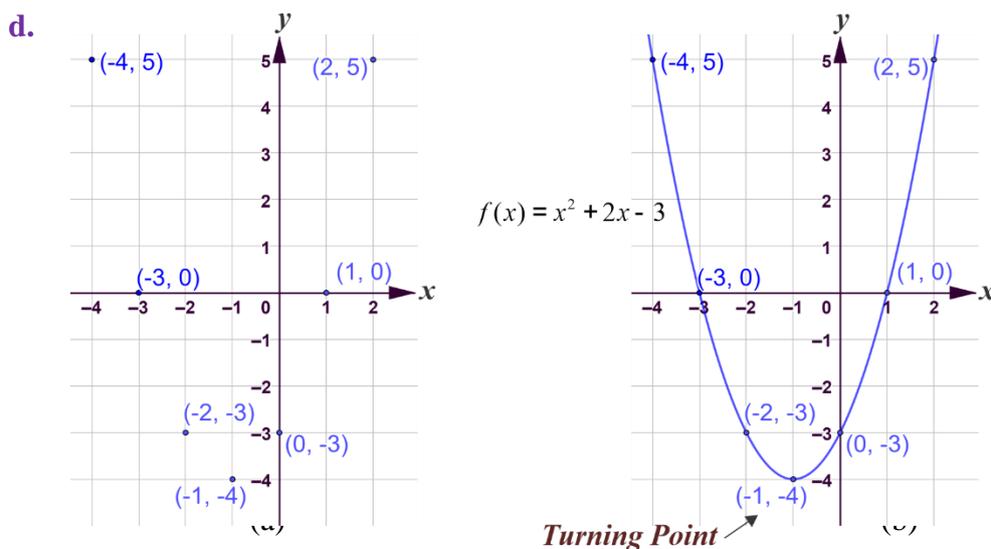
Figure 2.1

e. The domain is the set of all real numbers and the range is the set of all real numbers greater than or equal to -4.

The following are the properties of the quadratic polynomial $f(x) = ax^2 + bx + c$ when the leading coefficient $a > 0$.

    a. The degree is even.

    b. The maximum number of x-intercept is 2.

    c. The y-intercept is c.

    d. When $|x|$ is large (far to the right and far to the left) the graph of f goes upward. It is called upward parabola.

    e. The graph is smooth (has no sharp corner) and continuous (has no jump or hole).

    f. The domain of f is the set of all real numbers.

    g. The rang of $f(x)$ depends on the minimum value of f.

Example 2

Let $f(x) = -x^2 - 6x - 8$.

    a. Find the intercepts.

    b. Using completing the square method, rewrite f as $f(x) = 1 - (x + 3)^2$ and find the turning point.

    c. Complete the table of values below

x	-4	-3	-2	-1	0
$y = f(x)$					

    d. Sketch the graph of f, first by plotting the points (x, y) and then joining them by a smooth curve (**a smooth curve is a curve that has no sharp corner**).

    e. Find the domain and range of f.

Solution:

    a. $f(x) = y = -x^2 - 6x - 8 = (x + 2)(-x - 4)$. By making $f(x) = 0$ and solving the equation $-x^2 - 6x - 8 = (x + 2)(-x - 4) = 0$, we get $x = -2$ or $x = -4$.

By making $x = 0$ we have $y = -8$. Thus, $x = -2$ and $x = -4$ are the x-intercepts and $y = -8$ is the y-intercept.

    b. $y = f(x) = -x^2 - 6x - 8 = -(x^2 + 6x) - 8$

$$= -(x^2 + 6x + 9) + 9 - 8$$
$$= -(x + 3)^2 + 1$$
$$= 1 - (x + 3)^2$$

Since $(x + 3)^2 \geq 0$, $f(x) \leq 1$ for all values of x. Therefore, 1 is the maximum value of f. This maximum value of f is attained when $x = -3$. The point $(-3, 1)$ is called turning point for the graph of f.

c.

x	-4	-3	-2	-1	0
$y = f(x)$	0	1	0	-3	-8

d.
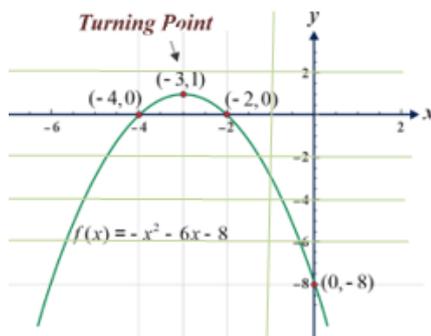
Figure 2.2

e. The domain is the set of all real numbers and the range is the set of all real numbers less than or equal to 1.

The following are the properties of the quadratic polynomial $f(x) = ax^2 + bx + c$ when the leading coefficient $a < 0$.

a. The degree is even.

b. The maximum number of x-intercept is 2.

c. The y-intercept is c.

d. When $|x|$ is large (far to the right and far to the left) the graph of f goes downward. It is called and downward parabola.

e. The graph is smooth (has no sharp corner) and continuous (has no jump or hole).

f. The domain of f is the set of all real numbers.

g. The rang of $f(x)$ depends on the maximum value of f.

> 1. Graph of a polynomial function is a smooth curve (has no sharp corner).
>
> 2. Functions whose graphs are not continuous and have sharp corners are not polynomial functions.

The absolute value function $f(x) = |x|$ is not a polynomial function. Because it has a sharp corner at the point (0, 0) as shown by figure 2.3 (a).

(a)   (b)   (c)
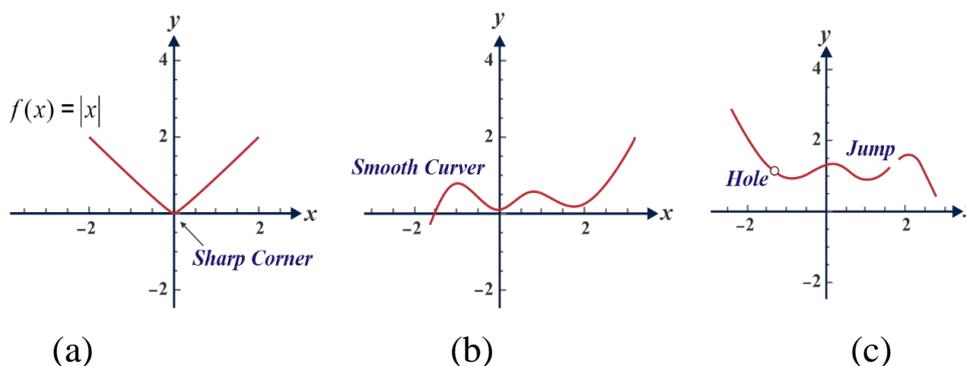
Figure 2.3

The function shown in figure 2.3 (c) is not polynomial because it is not continuous function. It has a hole and a jump.

Dear student, to study more on the property of graphs of polynomial functions, we will now try to observe the graphs of polynomial functions of higher degree that is when the degree $n \geq 3$.

Example 3

By sketching the graphs of $f(x) = x^3 - 1$ and $g(x) = -x^3 + 1$, describe the behavior of the graphs for large $|x|$.

Solution:

For $f(x) = x^3 - 1 = (x - 1)(x^2 + x + 1)$, the x-intercept is 1, the y-intercept is -1 and by finding some points that lie on the graph of f as shown by the table of values.

> Can you show that $(x - 1)(x^2 + x + 1) = x^3 - 1$?
>
> Can you show that $x^2 + x + 1 = 0$ has no real root?

x	-20	-10	-1	0	1	1.5	10	20
y	-8001	-1001	-2	-1	0	2.375	999	7999

We plot the points $(-1, -2)$, $(0, 1)$, and $\left(1.5, \frac{19}{8}\right)$ and connect them by a smooth curve and we use the other points to see the direction of the graph far to the right and far to the left along the x- axis.

For $g(x) = -x^3 + 1 = (-x + 1)(x^2 + x + 1)$, the x-intercept is 1, the y-intercept is 1 by finding some points that lie on the graph of g as shown by the table of values.

x	-20	-10	-1	0	1	1.5	10	20
y	8001	1001	2	1	0	-2.375	-999	-7999

Similarly, we plot the points $(-1, 2)$, $(0, 1)$ and $(1.5, -\frac{19}{8})$ and connect them by a smooth curve and we use the other points to see the direction of the graph far to the right and far to the left along the x-axis. The graphs of the two functions are:

a

b
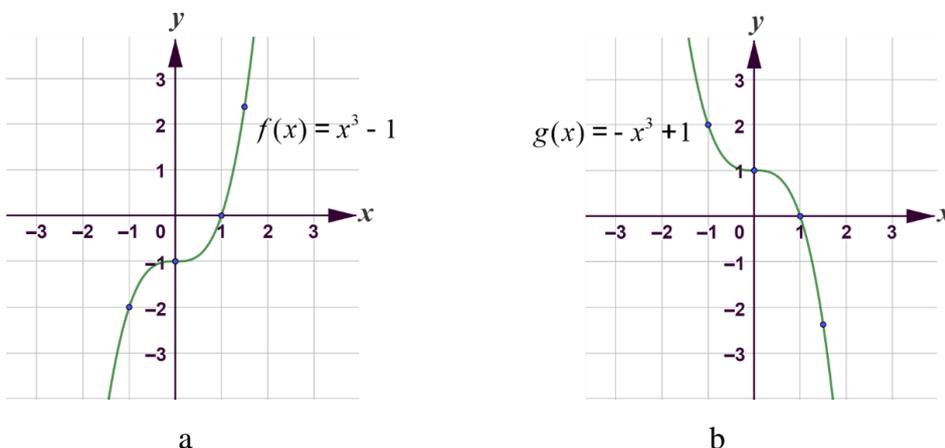
Figure 2.4

1. The degree of $f(x)$ is odd and its leading coefficient is positive. As shown in figure 2.4a, when x is large positive, $f(x)$ becomes large positive and the graph moves upwards and when x is large negative, $f(x)$ becomes large negative and the graph moves downwards.

2. The degree of $g(x)$ is odd and its leading coefficient is negative. As shown in figure 2.4b, when x is large positive, $f(x)$ becomes large negative and the graph moves downwards and when x is large negative, $f(x)$ becomes large positive and the graph moves upwards.

Example 4

Find the x-intercepts of the following polynomial functions.

a. $f(x) = (x - 1)^3$

b. $f(x) = (x + 1)^2(x + 2)$

c. $f(x) = (x + 1)(x - 2)(x - 3)$

Solution:

a. When you expand $(x - 1)^3$ you can see that $f(x) = x^3 - 3x^2 + 3x - 1$ which is a polynomial of degree three.

   To find the x-intercept we make $f(x) = 0$ and form the equation $(x - 1)^3 = 0$. Solving for x we have $x = 1$. That is $f(x)$ has only one x- intercept.

b. When you expand $(x + 1)^2(x + 2)$ you can see that $f(x) = x^3 + 4x^2 + 5x + 2$ which is a polynomial of degree three.

   To find the x-intercept we make $f(x) = 0$ and form the equation $(x + 1)^2(x + 2) = 0$.
   Solving for x we have $x = -1, x = -2$. That is $f(x)$ has two x- intercepts.

c. By expanding $(x + 1)(x - 2)(x - 3)$ you can see that $f(x) = x^3 - 4x^2 + x + 6$ which is a polynomial of degree three. To find the x-intercept we make $f(x) = 0$ and form the equation $(x + 1)(x - 2)(x - 3) = 0$.
   Solving for x we have $x = -1, x = 2$ and $x = 3$. That is $f(x)$ has two x- intercepts.

Example 5

Determine the behavior of the graphs of the following functions when $|x|$ is large. That is when the value of x is large positive number and large negative number.

   a. $f(x) = x^3 - 3x^2 - x + 2$ b. $g(x) = -x^3 + 2x^2 + 4x - 3$

Solution:

   a. $f(x) = x^3 - 3x^2 - x + 2$

The degree is 3 and the leading coefficient is positive.

When the value of x becomes large positive the value of $f(x)$ is dominated by the positive value of x^3 and $f(x)$ becomes large positive and its graph moves upward.

The following table shows some values of $f(x)$ for some appropriate positive values of x.

x	6	7	8	9	10
$f(x)$	104	191	314	479	692

When the value of x becomes large but negative the value of $f(x)$ is dominated by the value of x^3 which is negative. Because of this the value of $f(x)$ becomes large negative and its graph moves downward.

The following table shows some values of $f(x)$ for some appropriate positive values of x.

x	-6	-7	-8	-9	-10
$f(x)$	-316	-481	-694	-961	-1288

Dear student, the following is the graph of $f(x) = x^3 - 3x^2 - x + 2$ drawn using graph sketching software called GeoGebra. This graph is put for you simply to see the behavior of the graph when $|x|$ is large.
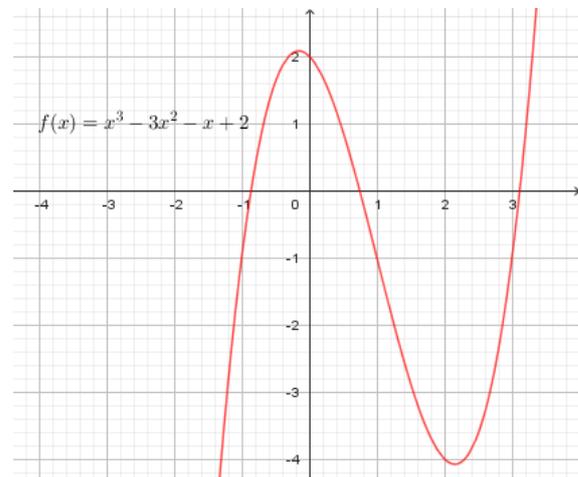
b. $g(x) = -x^3 + 2x^2 + 4x - 3$

The degree is 3 and the leading coefficient is negative.

Figure 2.5

When the value of x becomes large positive the value of $f(x)$ is dominated by the value of $-x^3$ which is negative and $f(x)$ becomes large negative and its graph moves downward.

The following table shows some values of $f(x)$ for some appropriate positive values of x.

x	4	5	6	7	8
$g(x)$	-19	-58	-123	-220	-355

When the value of x becomes large but negative the value of $f(x)$ is dominated by the value of $-x^3$ which is positive. Because of this the value of $f(x)$ becomes large positive and its graph moves upward.

The following table shows some values of $f(x)$ for some appropriate positive values of x.

x	-2	-3	-4	-5	-6
$f(x)$	5	30	77	152	261

Dear student, the following is the graph of $g(x) = -x^3 + 2x^2 + 4x - 3$ drawn using graph sketching software called GeoGebra. This graph is put for you simply to see the behavior of the graph when $|x|$ is large.
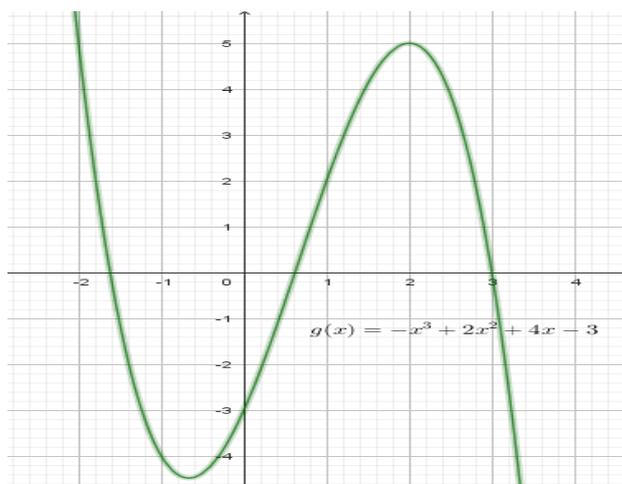
Figure 2.6

The following are the properties of the third-degree polynomial function
$$f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0,$$

    a.   The maximum number of intersections that the graph of f makes with the x-axis is 3.

    b.   The minimum number of intersections that the graph of f makes with the x-axis is 1.

    c.   When a_3 is positive and $|x|$ is large far to the right the graph moves upward and far to the left the graph moves downward.

    d.   When a_3 is negative and $|x|$ is large far to the right the graph moves downward and far to the left the graph moves upward.

    e.   The domain is the set of all real number.

    f.   The range is the set of all real numbers.

    g.   The graph is smooth and continuous?

Dear students, the following are examples of graphs of polynomial functions of degree four. The graphs are drawn using graph calculator software called GeoGebra. Observe the graphs carefully. You can see how the leading coefficient determine the behavior of the graph when the value of $|x|$ is large (far to the right and far to the left).
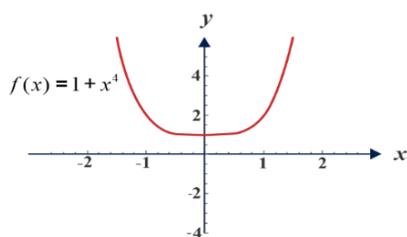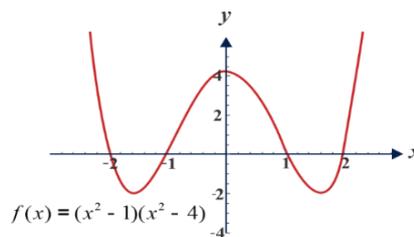
       a   b
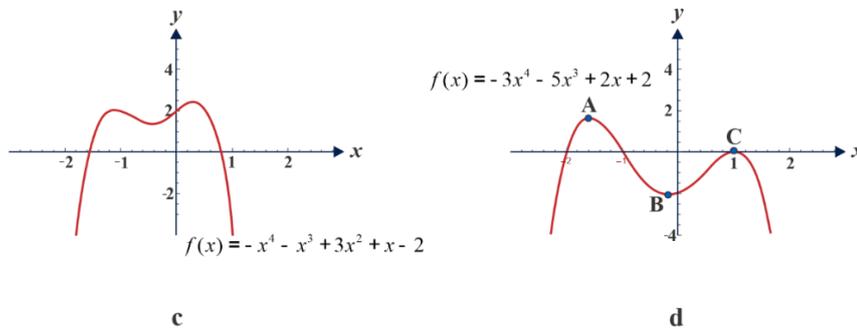
Figure 2.7

In figure 2.7a and figure 2.7b, the leading coefficient of $f(x)$ is positive and the degree is even. The values of $f(x)$ become large positive and the graphs go upward both far to the right and far to the left as the values of x become large in absolute value.

In figure 2.7c and figure 2.7d, the leading coefficient of $f(x)$ is negative and the degree is even. The values of $f(x)$ become large negative and the graphs go downward both far to the right and far to the left as the values x become large in absolute value.

From figure 2.7b, it is seen that the maximum number of intersections that the graph of a fourth-degree polynomial makes with the x-axis is 4 and the maximum number of turning points is 3.

As shown in figure 2.7a, the graph of a polynomial function of degree 4 may not intersect the x-axis. By applying the rational root test and the factor theorem, there is a possibility of finding the x-intercepts of a function. There is also a possibility of locating the real zeros using the location theorem.

With quadratic polynomials, we were able to algebraically find the maximum or minimum value of the function by finding the vertex (turning point). But for general polynomials, finding the turning points like A, B and C in figure 2.7d, is not possible without more advanced techniques from calculus (derivative of a function).

> ☞ The properties of the first and the third-degree polynomial are also applicable for polynomial functions of odd degree. The properties of the second and the fourth-degree polynomial are also applicable for polynomial functions of even degree.

Checklist

Dear learner, now it is time to check your understanding about this section. Read each question and put a mark (" ✓ ") in the box that is the appropriate choice to you.

	Yes	No
• Can you determine the minimum and maximum number of x-intercepts of the graph of a polynomial function of degree odd?	☐	☐

- Can you determine the minimum and maximum number of x-intercepts of the graph of a polynomial function of degree even? □ □

- Can you determine the properties of the graph of a polynomial function whose degree is odd, leading coefficient positive and when $|x|$ is large? □ □

- Can you determine the properties of the graph of a polynomial function whose degree is odd, leading coefficient negative and when $|x|$ is large? □ □

- Can you determine the properties of the graph of a polynomial function whose degree is even, leading coefficient positive and when $|x|$ is large? □ □

- Can you determine the properties of the graph of a polynomial function whose degree is even, leading coefficient negative and when $|x|$ is large? □ □

- Can you give an example of a polynomial function with degree even that does not intersect the x axis? □ □

If your response to the above check list questions is positive, proceed to do the self-test exercise given below and check your answer at the end of this module. If not please review the section again and again until you think you have clear understanding about it.

✖ Self-Test Exercise 5

1. For the function $f(x) = 2x^2 + 5x + 2,$

    a. Find the intercepts.

    b. Using completing the square method, rewrite f as $f(x) = -\frac{9}{8} + 2\left(x + \frac{5}{4}\right)^2$ and find the turning point.

    c. Complete the table of values below.

x	-3	-2	$-5/4$	-1	-0.5	0	1
$y = f(x)$	5	0	$-9/8$	-1	0	2	9

    d. Sketch the graph of f, first by plotting the points (x, y) and then joining them by a curve.

    e. Find the domain and range of f.

2. For the function $f(x) = -3x^2 + 4x,$

a. Find the intercepts.

b. Using completing the square method, rewrite f as $f(x) = \frac{4}{3} - 3\left(x - \frac{2}{3}\right)^2$ and find the turning point.

c. Complete the table of values below.

x	-0.5	0	$2/3$	1	$4/3$	2
$y = f(x)$						

d. Sketch the graph of f, first by plotting the points (x, y) and then joining them by a curve.

e. Find the domain and range of f.

3. Say True or False

    a. The graph of a polynomial function with degree odd always intersect the x-axis.

    b. The graph of a polynomial function with degree even may not intersect the x-axis.

    c. The graph of a polynomial function always intersects the y-axis.

    d. The graph of a polynomial function may contain a hole or a sharp corner.

    e. There is a polynomial function whose domain is not the set of all real numbers.

    f. The range of an even degree polynomial function is set of all real numbers.

4. For the polynomial functions given from a to d state the following properties of the graph of the functions without drawing their graphs.

    i. The behaviour of the graph as x takes values far to the right.

    ii. The behaviour of the graph as x takes values far to the left.

    iii. The maximum number of intersections with the x-axis.

        a. $f(x) = 2x^2 - 2x + 5$ **b.** $f(x) = 3 - 2x^4$

        c. $f(x) = -2x^3 + 4x - 5$ **d.** $f(x) = -3(x + 2)^2$

5. Graphs of some polynomial functions are given below. In each case identify the sign of the leading coefficient of the function. State whether the degree is even or odd.

a

b

c

d
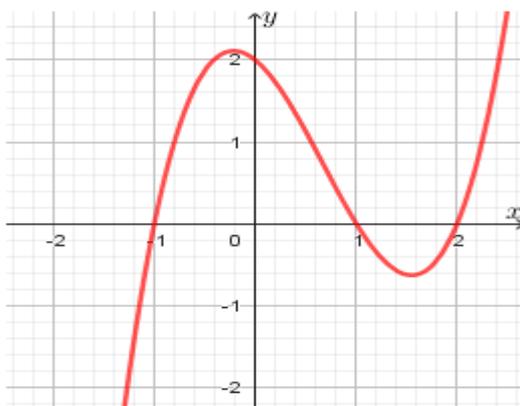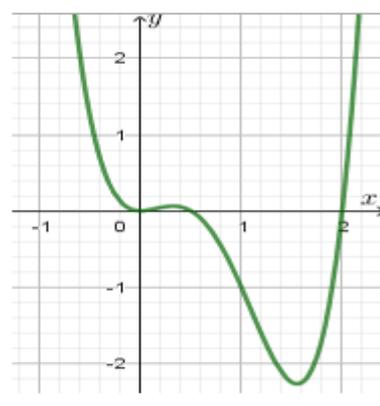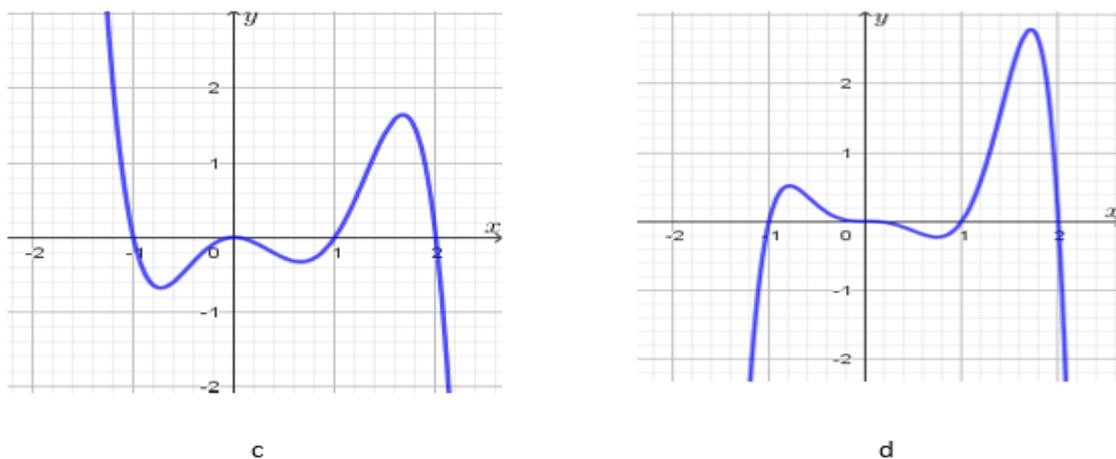
Figure 2.8

Section 6 | Applications

❀ Section Overview

Dear student, polynomials arise in problems involving areas and volumes. Here we are going to discuss polynomials in relation with the area of a rectangle and the volume of a box.

🎯 Section Objectives

By the end of this section, you will be able to:

- express area of a rectangle with a polynomial function.
- represent volume of a box with a polynomial function.

Example 1

A wire of length 56 m is bent into the shape of a rectangle. Find the maximum area it can be enclose and the dimensions of the rectangle of maximum area.
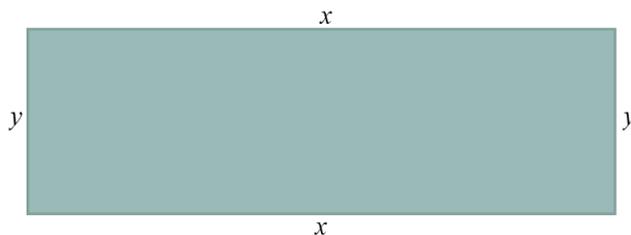
Figure 2.9

Solution:

Perimeter p of the rectangle,

$p = 2x + 2y$ . . . (i)

Area A of the rectangle,

$A = xy$ . . . (ii)

Since the rectangle is made by bending 56 meters wire, the perimeter of the rectangle is 56 m

$2x + 2y = 56$ ... (iii)

From (iii), solving for y (you can also solve for x).

$y = 28 - x$ . . . (iv)

Substituting the value of y in (iv) to the value of y in (ii) we get

$A = x(28 - x) = -x^2 + 28x$, and this is a quadratic polynomial.

Using completing the square method, we can rewrite the value A as in the following:

$A = -x^2 + 28x = -(x^2 - 28x) = -(x^2 - 28x + 196) + 196 = 196 - (x - 14)^2$

Thus, $A = 196 - (x - 14)^2$ and since $(x - 14)^2$ is always non-negative, we subtract positive number or zero from 196. This means the maximum value of A is 196. This maximum value is attained when x is 14. Substituting $x = 14$ into equation (iv) we get $y = 14$.

Therefore, the maximum area that can be enclosed is $196\ m^2$ and the dimension of the rectangle of maximum area is $x = 14\ m$ and $y = 14\ m$. That is when the rectangle is a square of side of length 14 m.

■ Example 2

Find the two real numbers whose difference is 16 and whose product is the minimum.

Solution:

Let the two numbers be x and y. Then,

$x - y = 16$ . . . (1)

If we denote the product of the two numbers by A, then

$A = xy$ . . . (2)

To find the minimum value of A, first we solve either for x or for y from equation (1).

Solving for y from equation (1) gives $y = x - 16$ and substituting this in equation (2) gives

$$A = x(x - 16) = x^2 - 16x$$
$$= (x^2 - 16x + 64) - 64$$
$$= (x - 8)^2 - 64 = -64 + (x - 8)^2$$

Since, $(x - 8)^2 \geq 0$

$A = -64 + (x - 8)^2 \geq -64$.

Therefore, -64 is the minimum product and this minimum product is obtained when the value of $x = 8$. And when $x = 8$, $y = x - 16 = -8$. That is, $y = -8$.

Checklist

Dear learner, now it is time to check your understanding about this section. Read each question and put a mark (" √ ") in the box that is the appropriate choice to you.

	Yes	No
• Given the perimeter of a rectangle, can you associate a polynomial function with the area of the rectangle?	□	□
• Can you relate the volume of a cube with a polynomial function	□	□

If your response to the above check list questions is positive, proceed to do the self-test exercise given below and check your answer at the end of this module. If not please review the section again and again until you think you have clear understanding about it.

Self-Test Exercise 6

1. A farmer has 80 meters of fencing wire to use to make a rectangular enclosure for hens as shown. He will leave an opening of 4 meters for the gate.
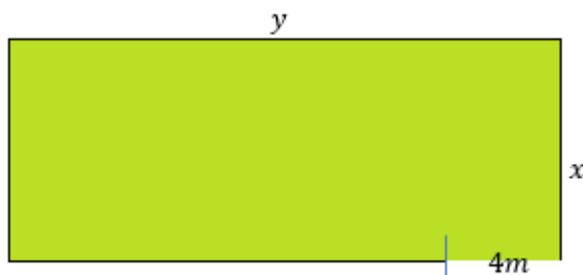
Figure 2.10

    a. Show that the area of the enclosure is given by $A = 42x - x^2$

    b. Find the value of x that will give maximum area.

    c. Calculate the maximum possible area.

2. A rectangular box is $2x$ units long, $2x + 3$ units wide and $3 - 2x$ units high.

    a. Express its volume as a polynomial.

    b. Find V when $x = 0.5$ cm.

    c. Can the value of x be 2 cm? Why?

Key terms

- ✓ Constant function
- ✓ Linear function
- ✓ Remainder theorem
- ✓ Constant term
- ✓ Location Theorem
- ✓ Turning point
- ✓ Degree
- ✓ Multiplicity
- ✓ x-intercept
- ✓ Domain
- ✓ Polynomial Division Theorem
- ✓ y-intercept
- ✓ Factor theorem
- ✓ Polynomial Function
- ✓ Zero(s) of polynomial
- ✓ Leading Coefficient
- ✓ Rational root test

Unit Summary

1. A linear function is given by $f(x) = ax + b,\ a \neq 0$.

2. A quadratic function is given by $f(x) = ax^2 + bx + c,\ a \neq 0$.

3. Let n be a non-negative integer and let $a_n, a_{n-1},\ \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$, the function $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ is called a polynomial function in x of degree n.

4. A polynomial function is over integers if its coefficients are all integers.

5. A polynomial function is over rational numbers if its coefficients are all rational numbers.

6. A polynomial function is over real numbers if its coefficients are all real numbers.

7. Operations on polynomial functions

     i.   Sum: $(f + g)(x) = f(x) + g(x)$

     ii.   Difference: $(f - g)(x) = f(x) - g(x)$

     iii.   Product: $(f \cdot g)(x) = f(x) \cdot g(x)$

     iv.   Quotient: $(f \div g)(x) = f(x) \div g(x),\ if\ g(x) \neq 0$

8. If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$ then there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = q(x)d(x) + r(x)$, where $r(x) = 0$ or the degree of $r(x)$ is less that the degree of $d(x)$.

9. If a polynomial $f(x)$ is divided by a first-degree polynomial of the form $x - c$ then the remainder is the number $f(c)$.

10. Given the polynomial function

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

If $p(c) = 0$, then c is the zero of the polynomial and a root of the equation $p(x) = 0$. Furthermore, $x - c$ is a factor of the polynomial.

11. For every polynomial function f and real numbers c, if $f(c) = 0$ then $x - c$ is a zero of the polynomial function f.

12. If the rational number $\frac{p}{q}$, in its lowest term, is a zero of the polynomial function

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ with integer coefficients, then p must be an integer factor of a_0 and q must be an integer factor of a_n.

13. Let a and b be real numbers such that $a < b$. If $f(x)$ is a polynomial function such that $f(a)$ and $f(b)$ have opposite signs, then there is at least one zero of $f(x)$ between a and b.

14. The graph of a polynomial function of degree n intersects the x -axis at most n times.

15. The graph of every polynomial function has no sharp corners; it is a smooth and continuous curve.

Review Exercises

I. Multiple Choice problems

1. Which of the following is not a polynomial function?

   A) $f(x) = 1 - x^2 + 23x^4$ **B)** $g(x) = x^3 + (x + 1)^2$

   C) $k(x) = x^{\frac{1}{2}} + 3$ **D)** $s(x) = 3\sqrt{2}^2 + 4$

2. If $f(x) = 4x^2 + \frac{5}{3}x^3 - \frac{2}{3}x + 1 - \frac{5x^3 + 3x^2 - x + 12}{3}$ is a polynomial function then which of the following is not true?

   A) Its degree is 2   **B)** The coefficient of x is $-\frac{2}{3}$

   C) Its leading coefficient is 3   **D)** Its constant term is -3

3. Given $f(x) = 2 + 3x - 2x^2 + x^3$, $g(x) = 2x^2 + 3x - 1$ and $h(x) = x - 1$, then $g - hf$ is

   A) $-3 + 2x + 7x^2 - 3x^3 + x^4$ **B)** $x^4 - 3x^3 + 5x^2 - x - 2$

   C) $x^4 - 3x^3 + 7x^2 - 2x - 1$ **D)** $-3x^3 + 7x^2 + 2x - 3$

4. If f and g are any two polynomials, then which of the following will always be a polynomial function?

   A) $f + g$ **B)** $f . g$ **C)** $\frac{f}{g}$ **D)** All except d

5. If $x + 1$ is a factor of $x^3 - kx^2 + 4x - 1$, the value of k is

   A) 6   **B)** -6   **C)** 3   **D)** 5

6. If $x + 1$ is a factor of $f(x) = ax^4 - 2kx^3 + ax^2 - kx + 2$ and $f(1) = 2$, then values of a and k are respectively,

   A) $a = -\frac{1}{2}$ and $k = -\frac{1}{3}$ **B)** $a = \frac{1}{2}$ and $k = \frac{1}{3}$

   C) $a = -\frac{1}{2}$ and $k = \frac{1}{3}$ **D)** $a = \frac{1}{2}$ and $k = -\frac{1}{3}$

7. Let g be a polynomial function in x. If the degree of $g(x)$ is greater than or equal to 1, then what

will be remainder when $f(x)$ is divided by $x + 2$?

   A. -2   **B.** $g(2)$

   C. $x - 2$ **D.** $g(-2)$

II. Short answer problems

1. In each of the following, find the quotient and the remainder when $p(x)$ is divided by $q(x)$

   a. $p(x) = 2x^3 - 4x^2 + 6x - 10$; $q(x) = x + 1$

   b. $p(x) = 5x^3 + 6x^2 - 4x - 15$; $q(x) = -x^2 + 1$

   c. $p(x) = 2x^5 + x^2 - 9$; $q(x) = x^3 - 2x$

   d. $p(x) = -x^4 + 2x^3 + 6x^2 - 2x - 11$: $q(x) = -x + 3$

2. Find the value of k such that,

   a. when $f(x) = x^3 - 2kx^2 + x - 8$ is divided by $x + 1$ it has a remainder of -3.

   b. $x - 2$ is a factor of $3x^3 + 2x^2 + kx - 8$.

3. Find the values of p and q if $x + 1$ is a common factor of $f(x) = 2x^4 - px^3 + qx^2 + 2$ and $g(x) = px^5 + 8x^3 - 4x^2 - qx$

4. Find a polynomial function f of degree 3 such that $f(2) = 48$ and $x + 1$, x and $x + 2$ are factors of the polynomial.

5. Find all the rational zeros of

   a. $f(x) = 6x^3 - 7x^2 - 9x - 2$

   b. $f(x) = 12x^4 - 22x^3 - 12x^2 + 33x - 9$

III. Workout problems

1. Given $x - 1$ is a factor of $p(x) = ax^3 - 6x^2 + bx + 6$ and -6 is the remainder when $p(x)$ is divided by $x - 2$. Find the values of a and b.

2. In each of the following, find a polynomial function f that has the given zeros satisfying the given condition.

   a. $1, 2, -3$ and $f(-2) = 4$ **b.** $0, -\frac{1}{3}, \frac{3}{2}, 3$ $f(1) = \frac{3}{4}$

3. Find all rational zeros of $f(x) = 8x^6 - 2x^4 + 4x^3 - 30x^2 + 8x + 12$.

4. Sketch the graphs of

   a. $f(x) = -2x^2 + 2x + 4$

   b. $f(x) = 3x^2 - 7x + 4$

5. Find the two real numbers whose difference is 22 and whose product is the minimum.

6. Find the two real numbers whose sum is 16 and whose product is the maximum.

UNIT 3

Exponential and Logarithmic Functions

❈ Introduction

Dear learner, two of the most important functions that occur in mathematics and its applications are the exponential function $f(x) = a^x$ and the logarithmic function $g(x) = \log_a x$. Such functions arise in many applications and are powerful mathematical tools for solving real life problems such as analyzing population growth and calculating compound interest in accounting, etc.

In this unit, we will examine their various properties and learn how they can be used in solving real life problems.

This unit is divided into 7 sections. In the first and the second sections, we shall define exponents and logarithms and discuss laws of exponents and logarithms.

In the third section, we will define exponential functions, draw their graphs and discuss the properties of the graphs.

The fourth section is about logarithmic functions and their graphs. In this section we will define logarithmic functions, draw their graphs and discuss the properties of the graphs.

The fifth section is about the relation between exponential and logarithmic functions with the same base. In this section we will discuss the relation of the two functions by sketching their graphs on the same xy-plane.

In the sixth section you will learn how to solve exponential and logarithmic equations. you will also see the importance of stating the universal set when we solve logarithmic equations.

In the last section you will look at applications of exponential and logarithmic functions. In this section you will learn how exponential and logarithmic functions are used in the computation of compound interest and population growth.

🎯 Learning Outcomes

At the end of this unit, you will be able to:

- ✳ apply the laws of exponents for real exponents.
- ✳ define exponential and logarithmic functions.
- ✳ identify domain and range of exponential and logarithmic functions.
- ✳ solve mathematical problems involving exponents and logarithms.

Unit contents

Section 1: Exponents

Section 2: Logarithms

Section 3: The Exponential Functions and Their Graphs

Section 4: The Logarithmic Functions and Their Graphs

Section 5: Relation Between Exponential and Logarithmic Functions with the same base

Section 6: Solving Exponential and Logarithmic Equations

Section 7: Applications

Unit Learning Strategies

We would like to mention the following for your study of this chapter.

- Create a schedule and follow it. Scheduling makes you more practical.
- Recalling the definitions, the laws and reading the properties given in the first three sections helps you to follow the applications you face in next sections easily. Do the activities and examples using the laws of exponents and logarithms until you master them and apply it to solve self-test exercises.
- Tables of logarithms are attached at the end of this module. You can use them to find the values of the common logarithm of a number. You can also use a scientific calculator for this purpose.
- Graph sketching application soft wares like GeoGebra can also be used to compare your sketching's in this unit.

When you attain what you had intended for a specific learning session, you should reward yourself with anything positive. This makes you more inspired to keep moving and cherish your learning and achievements.

Section 1 | Exponents

❋ Section Overview

Dear learner, in this section we will have a look at expressions like 7^3 or 2^5 or in general a^n. Base and exponent of this expression will be defined. We will also define different laws of exponent.

🎯 Section Objectives

By the end of this section, you will be able to: -

- explain what is meant by exponential expression.
- state the laws of exponents where the exponents are natural numbers.
- define zero and negative exponent.
- define rational exponent.

The symbol 2^5 means $\underbrace{2 \times 2 \times 2 \times 2 \times 2}_{5 \text{ factors}}$. This symbol is read as the 5th power of 2 'or 2 raised to 5'.

The expression 2^5 is just a shorthand way of writing the product of five twos. The number 2 is called the base, and 5 the exponent.

Similarly, if a is any real number, then a^4 stands for $a \times a \times a \times a$. Here a is the base, and 4 is the exponent.

Activity 3.1

1. Identify the base and the exponent of each of the following.

    a) 3^4 **b)** $(-3)^4$ **c)** $\left(\frac{3}{5}\right)^5$ **d)** $(-1)^9$

2. Find the values of each of the following.

    a) $(-1)^1$ **b)** $(-1)^4$ **c)** $\left(\frac{3}{5}\right)^1$ **d)** $(-2)^7$

    e) -2^4 **f)** $(-2)^4$ **g)** $\left(-\frac{2}{3}\right)^4$

Definition 3.1

👉 For a natural number n and a real number a, the expression a^n, read as "the nth power of a" or "a raised to n", is defined as follows:

$$a^n = \underbrace{a \times a \times \ldots \times a}_{n \text{ factors}}.$$

In a^n, a is called the base, n is called the exponent.

Special names are used when the exponent is 2 or 3. The expression b^2 is usually read as 'b squared', and the expression b^3 as 'b cubed'. Thus, 'two cubed' means $2^3 = 2 \times 2 \times 2 = 8$.

Note that, in $(-a)^n$ the base is $-a$ but in $-a^n$ the base is only a.

For example,

$(-3)^2 = (-3) \times (-3) = 9$ but $-3^2 = -(3 \times 3) = -9$.

$(3a)^3 = 3a \times 3a \times 3a = 27a^3$ but $3a^3 = 3 \times a \times a \times a = 3a^3$.

Example 1

Evaluate the following.

    a. 2^3 **b.** -2^3 **c.** $(-2)^3$ **d.** $-(-2)^3$ **e.** $(-4t)^3$

Solution:

   a) $2^3 = 2 \times 2 \times 2 = 8$

   b) $-2^3 = -(2 \times 2 \times 2) = -8$

   c) $(-2)^3 = (-2) \times (-2) \times (-2) = -8$

   d) $-(-2)^3 = -[(-2) \times (-2) \times (-2)] = -(-8) = 8$

   e) $(-4t)^3 = -4t \times -4t \times -4t = -64t^3$

Laws of Exponents

Dear student, in definition 3.1 you saw that if a is any real number and n is a positive integer then a^n means $\underbrace{a \times a \times a \times \ldots \times a}_{n \text{ factors}}$. The laws for the behaviors of exponents follow naturally from this meaning of a^n for a natural number n. Now, we are going to discuss on the five laws of exponents.

If a is a real number and m and n are natural numbers, then

$$a^m \times a^n = \underbrace{a \times a \times \ldots \times a}_{m \text{ factors}} \times \underbrace{a \times a \times \ldots \times a}_{n \text{ factors}}$$

$$= \underbrace{a \times a \times a \times \ldots \times a \times a \times a \times \ldots \times a}_{m+n \text{ factors}}$$

$$= a^{m+n}.$$

> **Law 1.** $a^m \times a^n = a^{m+n}$, where a is a real number and m and n are natural numbers.
>
> That is, to multiply two numbers in exponential form, with the same base, we add their exponents.

Example 2

$2^3 \times 2^4 = (2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2)$

$\qquad = \underbrace{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}_{7 \text{ factors}}$

$$= 2^7$$
$$= 2^{3+4}.$$

If a is a real number different from zero and m and n are natural numbers with $n > m$, then

$$\frac{a^n}{a^m} = \frac{\underbrace{a \times a \times a \times a \times \ldots \times a}_{n \text{ factors}}}{\underbrace{a \times a \times a \times \ldots \times a}_{m \text{ factors}}} = \frac{\overbrace{a \times a \times a \times \ldots \times a}^{m \text{ factors}} \times \overbrace{a \times a \times \ldots \times a}^{n-m \text{ factors}}}{\underbrace{a \times a \times a \times \ldots \times a}_{m \text{ factors}}}$$

$$= \frac{\underbrace{a \times a \times a \times \ldots \times a}_{m \text{ factors}}}{\underbrace{a \times a \times a \times \ldots \times a}_{m \text{ factors}}} \times \underbrace{a \times a \times \ldots \times a}_{n-m \text{ factors}}$$

$$= \underbrace{a \times a \times \ldots \times a}_{n-m \text{ factors}} = a^{n-m}.$$

☞ **Law 2.** $\frac{a^n}{a^m} = a^{n-m}$, where a is a real number different from zero and m and n are natural numbers.

Dear students, since we have not yet given any meaning to zero and negative exponents, n must be greater than m for law 2 to make sense.

▮ Example 3

$$\frac{3^6}{3^3} = \frac{3 \times 3 \times 3 \times 3 \times 3 \times 3}{3 \times 3 \times 3}$$

$$= 3 \times 3 \times 3 \times \frac{3 \times 3 \times 3}{3 \times 3 \times 3}$$

$$= 3 \times 3 \times 3$$

$$= 3^3 = 3^{6-3}$$

☞ **Law 3.** $(a^m)^n = a^{m \times n}$, where a is a real number and m and n are natural numbers.

(?) Can you prove Law 3 based on the definition of exponent?

▮ Example 4

$$(3^2)^3 = (3 \times 3) \times (3 \times 3) \times (3 \times 3)$$

$$= 3^6 = 3^{2 \times 3}.$$

From the definition of exponents, we know that if n is a natural number, then

$$(ab)^n = \underbrace{(ab) \times (ab) \times \ldots \times (ab)}_{n \text{ factors}}$$

$$= \underbrace{a \times a \times \ldots \times a}_{n \text{ factors}} \times \underbrace{b \times b \times \ldots \times b}_{n \text{ factors}}$$

$$= a^n b^n$$

👉 | **Law 4.** $(ab)^n = a^n b^n$, where a and b are real numbers and n is a natural number.

Example 5

$$(2 \times 3)^3 = (2 \times 3) \times (2 \times 3) \times (2 \times 3)$$
$$= (2 \times 2 \times 2) \times (3 \times 3 \times 3)$$
$$= 2^3 \times 3^3$$

If a and b are real numbers, $b \neq 0$ and n is a natural number, then by the definition of exponent,

$$\left(\frac{a}{b}\right)^n = \underbrace{\frac{a}{b} \times \frac{a}{b} \times \ldots \times \frac{a}{b}}_{n \text{ factors}} = \frac{\overbrace{a \times a \times \times \ldots \times a}^{n \text{ factors}}}{\underbrace{b \times b \times \ldots \times b}_{n \text{ factors}}} = \frac{a^n}{b^n}$$

👉 | **Law 5.** $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, where a and b are real numbers, $b \neq 0$ and n is a natural number.

Example 6

$$\left(\frac{2}{3}\right)^4 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{2 \times 2 \times 2 \times 2}{3 \times 3 \times 3 \times 3} = \frac{2^4}{3^4}$$

Zero and Negative Exponents

Activity 3.2

Evaluate each of the following using the law $\frac{a^m}{a^n} = a^{m-n}$.

a) $\frac{3^2}{3^2}$; what is the value of 3^0 ?

b) $\frac{(-3)^2}{(-3)^2}$; what is the value of $(-3)^0$?

c) $\frac{(0.1)^2}{(0.1)^2}$; what is the value of $(0.1)^0$?

Dear student, let us begin by extending a^n to include an exponent equal to 0 $(n = 0)$. We want to make sense of the expression a^0 in such a way that Laws 1, 2 and 3 hold. What happens to law 2 when $m = n$? Law 2 gives,

$$\frac{a^n}{a^n} = a^{n-n}$$
$$1 = a^0$$

It doesn't make sense to talk about a number being multiplied by itself 0 times. However, if we want law 2 to continue to be valid when $n = m$ then we must define the expression a^0 to mean the number 1.

☞ | If $a \neq 0$ then we define a^0 to be equal to 1. We do not attempt to give any meaning to the expression 0^0. It remains undefined.

Using this definition, we can check those laws 1 and 2 also remain valid.

That is, $a^m \times a^0 = a^m \times 1 = a^m = a^{m+0}$ and $\dfrac{a^n}{a^0} = \dfrac{a^n}{1} = a^n = a^{n-0}$.

To come up with a suitable meaning for negative exponents, we can take $n < m$ in law 2. For example, let us try $n = 2$ and $m = 3$.

$\dfrac{a^2}{a^3} = a^{2-3} = a^{-1}$.

But, $\dfrac{a^2}{a^3} = \dfrac{a \times a}{a \times a \times a} = \dfrac{1}{a}$.

Therefore, $\dfrac{1}{a} = a^{-1}$.

Similarly, for $a \neq 0$, $\dfrac{a^5}{a^8} = a^{-3}$ implies $\dfrac{1}{a^3} = a^{-3}$ and $\dfrac{a^8}{a^{20}} = a^{-12}$ implies $\dfrac{1}{a^{12}} = a^{-12}$.

Definition 3.2

☞ If n is a positive integer and $a \neq 0$, then

1) $a^0 = 1$ and 0^0 is undefined.

2) $a^{-n} = \dfrac{1}{a^n}$.

Example 7

Evaluate the following.

a) 1^0 b) $(-10)^0$ c) $\left(\dfrac{1}{2}\right)^0$ d) $(0.123)^0$

Solution:

a) $1^0 = 1$ b) $(-10)^0 = 1$ c) $\left(\dfrac{1}{2}\right)^0 = 1$ d) $(0.123)^0 = 1$

Example 8

Evaluate the following.

a) 3^{-2} b) $\left(\dfrac{2}{3}\right)^{-3}$ c) $\left(\dfrac{1}{2}\right)^{-2}$

Solution:

a) $3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$

b) $\left(\frac{2}{3}\right)^{-3} = \frac{1}{\left(\frac{2}{3}\right)^3} = \frac{1}{\frac{2^3}{3^3}} = \frac{3^3}{2^3} = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$

c) $\left(\frac{1}{2}\right)^{-2} = \frac{1}{\left(\frac{1}{2}\right)^2} = \frac{1}{\frac{1^2}{2^2}} = \frac{2^2}{1^2} = 2^2 = 4$

Dear student, the following is the generalization of the laws for integer exponents.

For real numbers a and b and integers m and n, the following laws of exponents hold true.

1. $a^m \times a^n = a^{m+n}$ . . . law of multiplication of powers of the same base.

2. $\frac{a^m}{a^n} = a^{m-n}$ . . . law of division of powers of the same base.

3. $(a^m)^n = a^{mn}$ . . . law of power of a power.

4. $(a \times b)^n = a^n \times b^n$ . . . law of a power of a product.

5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ . . . law of a power of a quotient.

Example 9

Simplify each of the following.

a) $a^{-2} \times a^5$ **b)** $(3a)^4 \times (3a)^{-2}$ **c)** $\frac{5^4}{5^6}$ **d)** $(a^2)^{-3}$

e) $x^2 \times x^{-3} \times x^4$ **f)** $\left(\frac{3a}{16}\right)^2$ **g)** $(81)^t \times 9^{2t}$ **h)** $(-5x \times 3y)^2$

Solution:

a) $a^{-2} \times a^5 = a^{-2+5} = a^3$

b) $(3a)^4 \times (3a)^{-2} = (3a)^{4+(-2)} = (3a)^2 = 3^2 a^2 = 9a^2$

c) $\frac{5^4}{5^6} = 5^{4-6} = 5^{-2} = \frac{1}{25}$

d) $(a^2)^{-3} = a^{2 \times (-3)} = a^{-6} = \frac{1}{a^6}$

e) $x^2 \times x^{-3} \times x^4 = (x^2 \times x^{-3}) \times x^4 = x^{-1} \times x^4 = x^{-1+4} = x^3$

f) $\left(\frac{3a}{16}\right)^2 = \frac{3^2 \times a^2}{(16)^2} = \frac{9a^2}{256}$

g) $(81)^t \times 9^{2t} = 3^{4t} \times (3^2)^{2t} = 3^{4t} \times 3^{4t} = 3^{4t+4t} = 3^{8t}$

h) $(-5x \times 3y)^2 = (-5x)^2 \times (3y)^2 = (-5)^2 x^2 \times 3^2 y^2 = 25x^2 \times 9y^2 = 225x^2 y^2$

The Rational Exponent

Dear student, we can extend the definition of exponents even further to include rational numbers. For example, to define powers like $a^{\frac{1}{2}}$, consider $9^{\frac{1}{2}}$.

Applying law 3 and taking the square of $9^{\frac{1}{2}}$, we get $\left(9^{\frac{1}{2}}\right)^2 = 9^{\left(\frac{1}{2}\right)\cdot(2)} = 9^1 = 9$. Thus, $9^{\frac{1}{2}}$ is a number that yields 9 when squared. There are two numbers whose square is 9. They are 3 and -3. We define

$9^{\frac{1}{2}}$ to be the positive square root of 9. That is, 3. To avoid ambiguity, we define $a^{\frac{1}{2}}$ as the non-negative number that yields a when squared. Thus, $9^{\frac{1}{2}} = 3$.

☞ In general, $a^{\frac{1}{2}}$ is defined to be the positive square root of a, which can also be written as $\sqrt{a}$. So $a^{\frac{1}{2}} = \sqrt{a}$.

Of course, a must be positive if $a^{\frac{1}{2}}$ is meaningful because if we take any real number and multiply it by itself, then we get a positive number.

We can arrive at the definition of $a^{\frac{1}{3}}$ in the same way as we did for $a^{\frac{1}{2}}$. For example, if we cube $8^{\frac{1}{3}}$, we get $\left(8^{\frac{1}{3}}\right)^{3} = (8)^{\left(\frac{1}{3}\times 3\right)} = 8$. Thus, $8^{\frac{1}{3}}$ is the number that yields 8 when cubed. Since $2^{3} = 8$ we have $8^{\frac{1}{3}} = 2$. Similarly, $(-27)^{\frac{1}{3}} = -3$.

This time we have no trouble giving a meaning to $(-27)^{\frac{1}{3}}$ even though $-27 < 0$. There is a number when multiplied by itself 3 times gives -27, namely -3, so $(-27)^{\frac{1}{3}} = -3$.

Thus, we define $a^{\frac{1}{3}}$ (called the cube root of) as the quantity that yields a when cubed.

> ## Definition 3.3 The Rational Exponent $a^{\frac{1}{n}}$
>
> ☞ If a is positive, then $a^{\frac{1}{n}}$ is defined to be a positive number whose n^{th} power is equal to a. This number is called the nth root of a and sometimes written as $\sqrt[n]{a}$.
>
> If n is even and a is negative, $a^{\frac{1}{n}}$ cannot be defined because raising any number to an even power result in a positive number.
>
> If n is odd and a is negative, $a^{\frac{1}{n}}$ can be defined. It is a negative number whose n^{th} power is equal to a.

Example 10

Express in the form $a^{\frac{1}{n}}$ and evaluate the following.

  a) $\sqrt[4]{16}$ b) $\sqrt[3]{8}$ c) $\sqrt[3]{-8}$ d) $\sqrt[4]{-16}$

Solution:

  a) Since $2^{4} = 16$, $\sqrt[4]{16} = (16)^{\frac{1}{4}} = 2$

  b) Since $2^{3} = 8$, $\sqrt[3]{8} = (8)^{\frac{1}{3}} = 2$

  c) Since $(-2)^{3} = -8$, $\sqrt[3]{-8} = (-8)^{\frac{1}{3}} = -2$

  d) $\sqrt[4]{-16} = (-16)^{\frac{1}{4}}$ is not a real number because there is no real number a such that a^{4} is -16

Activity 3.3

Simplify the following expressions.

    a) $2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ **b)** $\sqrt{2} \times \sqrt{2}$ **c)** $2^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ **d)** $\sqrt{2} \times \sqrt{2} \times \sqrt{2}$

Dear student, so far, we have defined $a^{\frac{1}{n}}$, where n is a natural number. With the help of the third law for exponent, we can notice that,

$$\frac{m}{n} = m \times \frac{1}{n}.$$

So, if law 3 is to hold then

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^{m}.$$

Therefore, we can define the expression $a^{\frac{m}{n}}$, where m and n are natural numbers and $\frac{m}{n}$ is reduced to lowest term as in definition 3.4.

> ### Definition 3.4 The Rational Exponent $a^{\frac{m}{n}}$
>
> If $a^{\frac{1}{n}}$ is a real number, then $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^{m}$ (that is, the nth root of a raised to the mth power).
>
> We can also define negative rational exponents:
>
> $$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} \ \ (a \neq 0)$$

> Note: The laws of exponents discussed earlier for integral exponents hold true for rational exponents.

Example 11

Evaluate the following expressions.

a) $4^{\frac{1}{3}} \times 16^{\frac{1}{3}}$ **b)** $\dfrac{3^{\frac{1}{2}}}{27^{\frac{3}{2}}}$ **c)** $27^{\frac{2}{3}}$ **d)** $\left(3m^{\frac{1}{2}} \times 4n^{\frac{3}{2}}\right)^{2}$ **e)** $(-32)^{-\frac{3}{5}}$

Solution:

a) $4^{\frac{1}{3}} \times 16^{\frac{1}{3}} = (2^2)^{\frac{1}{3}} \times (2^4)^{\frac{1}{3}} = 2^{\frac{2}{3}} \times 2^{\frac{4}{3}} = 2^{\frac{2}{3}+\frac{4}{3}} = 2^2 = 4$

b) $\dfrac{3^{\frac{1}{2}}}{27^{\frac{3}{2}}} = \dfrac{3^{\frac{1}{2}}}{3^{\frac{9}{2}}} = 3^{\frac{1}{2}-\frac{9}{2}} = 3^{-\frac{8}{2}} = 3^{-4} = \dfrac{1}{3^4} = \dfrac{1}{81}$

c) $27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^{2} = 3^2 = 9$

d) $\left(3m^{\frac{1}{2}} \times 4n^{\frac{3}{2}}\right)^2 = \left(3m^{\frac{1}{2}}\right)^2 \times \left(4n^{\frac{3}{2}}\right)^2 = 3^2 m \times 4^2 n^3 = 9m \times 16n^3 = 144mn^3$

e) $(-32)^{-\frac{3}{5}} = \dfrac{1}{(-32)^{\frac{3}{5}}} = \dfrac{1}{\left((-32)^{\frac{1}{5}}\right)^3} = \dfrac{1}{(-2)^3} = -\dfrac{1}{8}$

Definition 3.5 shows the radical notation is an alternative way of writing an expression with rational exponents.

> ## Definition 3.5
>
> ☞
>
> If $a^{\frac{1}{n}}$ is a real number and m an integer then
> $$a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m} \quad (or\ a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m).$$

Example 12

Express in the form $a^{\frac{m}{n}}$, with a being a prime number.

a) $\sqrt[3]{4}$ **b)** $\sqrt[5]{27}$ **c)** $\dfrac{\sqrt[3]{8}}{\sqrt[3]{4}}$ **d)** $\left(\sqrt[3]{625}\right)^2$ **e)** $\sqrt[3]{\sqrt[2]{64}}$

Solution:

a) $\sqrt[3]{4} = 4^{\frac{1}{3}} = (2^2)^{\frac{1}{3}} = 2^{\frac{2}{3}}.$

b) $\sqrt[5]{27} = (27)^{\frac{1}{5}} = (3^3)^{\frac{1}{5}} = 3^{\frac{3}{5}}$

c) $\dfrac{\sqrt[3]{8}}{\sqrt[3]{4}} = \dfrac{(2^3)^{\frac{1}{3}}}{(2^2)^{,\frac{1}{3}}} = \dfrac{2^1}{2^{\frac{2}{3}}} = 2^{1-\frac{2}{3}} = 2^{\frac{1}{3}}$

d) $\left(\sqrt[3]{625}\right)^2 = \left((5^4)^{\frac{1}{3}}\right)^2 = \left(5^{\frac{4}{3}}\right)^2 = 5^{\frac{8}{3}}$

e) $\sqrt[3]{\sqrt[2]{64}} = \sqrt[3]{(64)^{\frac{1}{2}}} = \sqrt[3]{(8^2)^{\frac{1}{2}}} = \sqrt[3]{8} = \sqrt[3]{2^3} = (2^3)^{\frac{1}{3}} = 2$

Irrational Exponents

Dear student, the expressions $3^{\sqrt{2}}$, $2^{2\sqrt{3}}$, 5^{π} are powers with irrational exponents. Because $\sqrt{2}$, $2\sqrt{3}$ and π are irrational numbers.

> **?** Can you find the value of $3^{\sqrt{2}}$?

Using calculator, the value of $\sqrt{2} = 1.41421356 \ldots$

Therefore, $3^{\sqrt{2}} = 3^{1.41421356 \ldots}$

It is not possible to calculate $3^{\sqrt{2}} = 3^{1.41421356 \cdots}$ because $1.41421356 \ldots$ has infinite decimals.

But one can approximate the value of $3^{1.41421356 \cdots}$ as follows.

To approximate $3^{1.41421356 \ldots}$

$3^1 = 3$

$3^{1.4} = 3^{\frac{14}{10}} = 4.65553672$, to eight decimal places

$\quad\quad$ ($3^{\frac{14}{10}}$ is an expression with rational exponent, use a calculator to find its value)

$3^{1.41} = 3^{\frac{141}{100}} = 4.70696500$

$3^{1.414} = 3^{\frac{1414}{1000}} = 4.72769503$

$3^{1.4142} = 4.72873393$

$3^{1.41421} = 4.72878588$

$3^{1.4142135} = 4.72880406$

$3^{1.41421356} = 4.72880437$

As we can see from the above list, the values of

$3^1, 3^{1.4}, 3^{1.41}, 3^{1.414}, 3^{1.4142}, 3^{1.41421}, 3^{1.4142135}, 3^{1.41421356}$

are approaching to some number. For example, the first six decimals of the values of $3^{1.4142135}$ and $3^{1.41421356}$ are the same. That is 4.728804.

By continuing the approximation of $3^{\sqrt{2}}$ like above we can say $3^{\sqrt{2}} \cong 4.72880437$ to eight decimal places.

Now, if we define 3^x then $3^{\sqrt{2}}$ is the real number that 3^x approaches when x gets closer and closer to $\sqrt{2}$.

In general, if we define a^x and b is an irrational number, then a^b is a real number that a^x approaches when x gets closer and closer to b.

The above statement about irrational exponents suggests that the expression a^x is defined not only for integral and rational exponents but also for irrational exponents.

The laws of exponents discussed earlier for integral and rational exponents continue to hold true for irrational exponents.

Example 13

Simplify each of the following.

$\quad$ a) $\ 3^{\sqrt{2}} \times 3^{\sqrt{2}}$ $\quad\quad\quad\quad$ b) $\left(4^{\sqrt{2}}\right)^3$ $\quad\quad\quad\quad$ c) $\left(3^{\sqrt{2}}\right)^{\sqrt{2}}$

$\quad$ d) $\dfrac{2^{\sqrt{3}+2}}{2^{\sqrt{3}-3}}$ $\quad\quad\quad\quad$ e) $5^{\sqrt{2}} \times 5^{\sqrt{3}}$ $\quad\quad\quad\quad$ f) $\left(5^{\sqrt{2}}\right)^{\sqrt{3}}$

Solution:

$\quad$ a) $\ 3^{\sqrt{2}} \times 3^{\sqrt{2}} = 3^{\sqrt{2}+\sqrt{2}} = 3^{2\sqrt{2}} = (3^2)^{\sqrt{2}} = 9^{\sqrt{2}}$

$\quad$ b) $\left(4^{\sqrt{2}}\right)^3 = 4^{3\times\sqrt{2}} = (4^3)^{\sqrt{2}} = 64^{\sqrt{2}}$

c) $\left(3^{\sqrt{2}}\right)^{\sqrt{2}} = 3^{\sqrt{2}\times\sqrt{2}} = 3^2 = 9$

d) $\dfrac{2^{\sqrt{3}+2}}{2^{\sqrt{3}-3}} = \dfrac{2^{\sqrt{3}}\times 2^2}{2^{\sqrt{3}}\times 2^{-3}} = \dfrac{2^2}{2^{-3}} = 2^{2-(-3)} = 2^5 = 32$ or $\dfrac{2^{\sqrt{3}+2}}{2^{\sqrt{3}-3}} = (2)^{(\sqrt{3}+2)-(\sqrt{3}-3)} = 2^5 = 32$

e) $5^{\sqrt{2}} \times 5^{\sqrt{3}} = 5^{\sqrt{2}+\sqrt{3}}$

f) $\left(5^{\sqrt{2}}\right)^{\sqrt{3}} = 5^{\sqrt{2}\times\sqrt{3}} = 5^{\sqrt{6}}$

☑ Checklist

Dear learner, now it is time to check your understanding about exponents. Read each question and put a mark (" ✔ ") in the box that is the appropriate choice to you.

	Yes	No
• Can you explain what is meant by exponential expression?	☐	☐
• Can you state the laws of exponents when the exponents are natural numbers?	☐	☐
• Can you define zero and negative exponent?	☐	☐
• Can you define the n$^{\text{th}}$ root of a number a?	☐	☐
• Can you define rational exponent?	☐	☐
• Can you state the laws of exponents when the exponents are real numbers?	☐	☐

If your response to the above check list questions is positive, proceed to do the self-test exercise given below and check your answer at the end of this module. If not, please review the section again and again until you think you have clear understanding about it.

✖ Self-Test Exercise 1

1. Evaluate the following.

a) 3^2 　　　　**b)** -3^2 　　　　**c)** $(-3)^2$ 　　　　**d)** $-(-4)^3$

2. Simplify the exponential expressions.

a) $a^4 \times a^3$ 　　**b)** $\dfrac{a^4}{a^2}$ 　　　**c)** $(a^4)^2$ 　　　**d)** $(2a)^3$ 　　　**e)** $\left(\dfrac{a}{2}\right)^3$

3. Evaluate the following.

a) 3^0 　　**b)** $(-10)^0$ 　　**c)** $\left(\dfrac{24}{59}\right)^0$ 　　**d)** 2^{-2} 　　**e)** 10^{-2} 　　**f)** $\left(\dfrac{2}{3}\right)^{-3}$ 　　**g)** $\left(\dfrac{1}{2}\right)^{-5}$

4. Simplify the exponential expressions using laws of exponents.

　　a) $x^{-2} \times x^3$ 　　　　　**b)** $\dfrac{2^4}{2^7}$ 　　　　　**c)** $\dfrac{3^{-5}}{3^{-2}}$

d) $(-x \times 2y)^2$ **e)** $n^{-3} \times n^{-2}$ **f)** $(4m)^3 \times (8m)^{-2}$

g) $2^t \times 2^{3t} \times 2^{2t}$ **h)** $(2^3)^{2n}$ **i)** $(2a^{-3} \times b^2)^{-2}$

j) $\dfrac{(a^2)^{-2} \times (a^3)^3}{a^9}$ **k)** $\left(\dfrac{m^{-5}n^2}{n^{-2}m^6}\right)^{-1}$

5. Express in the form $a^{\frac{1}{n}}$ and evaluate the following.

a) $\sqrt[4]{81}$ **b)** $\sqrt[5]{\dfrac{1}{32}}$ **c)** $\sqrt[3]{-125}$ **d)** $\sqrt[3]{27}$ **e)** $-\sqrt[3]{-1000}$ **f)** $\sqrt[4]{-10000}$

6. Simplify each of the following expressions.

a) $(-27)^{-\frac{2}{3}}$ **b)** $3^{\frac{4}{3}} \times 9^{\frac{4}{3}}$ **c)** $\dfrac{81^{\frac{2}{5}}}{27^{\frac{6}{5}}}$ **d)** $\left(m^{\frac{1}{4}} \times 3n^{\frac{5}{2}}\right)^2$

e) $\left(\dfrac{a^{\frac{1}{3}}}{b^{-\frac{3}{2}}}\right)^{-6}$ **f)** $\dfrac{(x^2)^{-\frac{1}{4}} \times (x^3)^{\frac{2}{9}}}{x^{\frac{1}{2}}}$ **g)** $\left(\dfrac{x^{\frac{1}{8}} \, y^{-\frac{1}{3}}}{y^{\frac{2}{3}} \, x^{\frac{1}{4}}}\right)^4$

7. Express in the form $a^{\frac{m}{n}}$, with a being a prime number.

a) $\sqrt[5]{81}$ **b)** $\sqrt[3]{32}$ **c)** $\dfrac{\sqrt[4]{25}}{\sqrt[4]{5}}$ **d)** $\dfrac{\sqrt[3]{40}}{\sqrt[3]{5}}$

e) $\left(\sqrt[4]{27}\right)^6$ **f)** $\left(\sqrt[3]{121}\right)^2$

8. Simplify each of the following.

a) $3^{\sqrt{3}} \times 3^{\sqrt{3}}$ **b)** $\left(4^{\sqrt{2}}\right)^2$ **c)** $\left(\sqrt[3]{8}\right)^{-2}$ **d)** $\sqrt{2}^{\sqrt{3}} \times \sqrt{2}^{\sqrt{27}}$

e) $\dfrac{3^{\sqrt{2}+3}}{3^{\sqrt{2}-1}}$ **f)** $\left(2^{\sqrt{3}}\right)^{\sqrt{27}}$ **g)** $\dfrac{3^{\sqrt{2}} \times 9^{\sqrt{8}}}{27^{\sqrt{18}}}$ **h)** $\dfrac{(5^{\sqrt{3}})^2 \times 5^{-\sqrt{12}} \times 25^{\sqrt{3}}}{5^{\sqrt{27}}}$

Section 2 | Logarithms

❈ Section Overview

In this section, we will have a look at expressions like $\log_2 3$ or $\log_{10} 100$ or in general $\log_b a$. We will also define different laws of logarithms.

🎯 Section Objectives

By the end of this section, you will be able to: -

- explain what is meant by logarithmic expression.
- state the laws of logarithms.
- define common logarithm.
- use the table for finding logarithm of a given positive number and antilogarithm of a number.

Dear student, in the exponential equation $2^3 = 8,$ the base is 2 and the exponent is 3. We write this equation in logarithm form as $\log_2 8 = 3$. We read this as "the logarithm of 8 to the base 2 is 3".

The logarithm to base a of a number $x > 0$ written $\log_a x$ is that power to which a must be raised to obtain the number x. $\log_a x$ is called a logarithmic expression.

For example,

$\log_3 9 = 2$ because $3^2 = 9$.

$\log_3 \left(\frac{1}{9}\right) = -2$ because $3^{-2} = \frac{1}{9}$.

$\log_5 1 = 0$ because $5^0 = 1$.

Definition 3.6

For a $> 0, a \neq 1,$ and $c > 0$

$$\log_a c = b \text{ if and only if } a^b = c.$$

Dear student, by doing the following examples you can practice on how to change exponential statements to logarithm statements and logarithm statements to exponential statements. For this observe the relationship between logarithm and exponent stated in the definition 3.6 carefully.

Example 1

Convert each of the following to logarithmic statement.

   a) $2^5 = 32$ **b)** $3^4 = 81$ **c)** $4^{\frac{1}{2}} = 2$

Solution:

   a) From $2^5 = 32$, we have $\log_2 32 = 5$.

   b) From $3^4 = 81$, we have $\log_3 81 = 4$.

   c) From $4^{\frac{1}{2}} = 2$, we have $\log_4 2 = \frac{1}{2}$.

Example 2

Convert each of the following to exponential statement.

   a) $\log_2 128 = 7$ **b)** $\log_3 \left(\frac{1}{27}\right) = -3$ **c)** $\log_{10} \sqrt[3]{10} = \frac{1}{3}$

Solution:

   a) From $\log_2 128 = 7$, we have $2^7 = 128$.

   b) From $\log_3 \left(\frac{1}{27}\right) = -3$, we have $3^{-3} = \frac{1}{27}$.

   c) From $\log_{10} \sqrt[3]{10} = \frac{1}{3}$, we have $10^{\frac{1}{3}} = \sqrt[3]{10}$.

Example 3

Find the value of each of the following logarithms.

    a) $\log_2 64$ **b)** $\log_3 \left(\frac{1}{81}\right)$ **c)** $\log_{10} 100$ **d)** $\log_{10} 0.01$

Solution:

a) Since $64 = 2^6$, the exponent to which we raise 2 to get 64 is 6.

    So, $\log_2 64 = 6$.

b) Since $\frac{1}{81} = \frac{1}{3^4} = 3^{-4}$, the exponent to which we raise 3 to get $\frac{1}{81}$ is -4.

    Hence, $\log_3 \left(\frac{1}{81}\right) = -4$.

c) The exponent to which we raise 10 to get 100 is 2 as $10^2 = 100$. Therefore, $\log_{10} 100 = 2$.

d) Since $0.01 = \frac{1}{100} = \frac{1}{10^2} = 10^{-2}$, the exponent to which we raise 10 to get $\frac{1}{100}$ is -2.

    Thus, $\log_{10} 0.01 = -2$.

Properties of logarithms

Dear student, the following properties follow directly from the definition of the logarithm (definition 3.6). Observe the definition carefully to understand the properties.

For $a > 0$ and $a \neq 1$

1. $\log_a 1 = 0$ because $a^0 = 1$.

2. $\log_a a = 1$ because $a^1 = a$.

3. $\log_a a^p = p$ and $a^{\log_a p} = p$ ... Inverse property.

4. If $\log_a M = \log_a N$, then $M = N$ ... One-to-One property.

Example 4

Use properties of logarithms to answer the following questions.

    a) Find p such that $\log_2 p = \log_2 5$.   **b)** Simplify $\log_2 2^p$.

    c) Find p such that $\log_3 3 = p$.   **d)** Simplify $5^{\log_5 p}$.

Solution:

a) Using property 4, we can see that $p = 5$.

b) Using property 3, it follows that $\log_2 2^p = p$.

c) Using property 2, we can conclude that $p = 1$.

d) Using property 3, it follows that $5^{\log_5 p} = p$.

Laws of logarithms

Dear student, we now establish laws of logarithms. The laws are represented by theorems and we

Theorem 3.1 Logarithms of products

For any positive numbers M, N and $a > 0$ and $a \neq 1$,

$$\log_a MN = \log_a M + \log_a N$$

(The logarithm of a product is the sum of the logarithms of the factors.)

Proof:

Let $\log_a M = p$ and $\log_a N = q$.

(**?**) Do you remember definition 3.6 ? What is its importance?

Converting $\log_a M = p$ to is corresponding exponential equations, we get $a^p = M$.

Converting $\log_a N = q$ to is corresponding exponential equations, we get $a^q = N$.

Now, $MN = a^p \cdot a^q = a^{p+q}$.

This implies $MN = a^{p+q}$.

Converting back to a logarithmic equation, we obtain

$\log_a MN = p + q$.

But, $p = \log_a M$ and $q = \log_a N$.

Therefore, $\log_a MN = p + q = \log_a M + \log_a N$.

Example 5

Express $\log_2(4 \times 8)$ as a sum of logarithms.

Solution:

We have $\log_2(4 \times 8) = \log_2 4 + \log_2 8$ . . . using the product law

Example 6

Express $\log_3 5 + \log_3 8$ as a single logarithm.

Solution:

We have $\log_3 5 + \log_3 8 = \log_3(5 \times 8) = \log_3 40$ . . . using the product law

Theorem 3.2 Logarithms of powers

For any positive number M, any real number r, and $a > 0$ and $a \neq 1$,

$$\log_a(M)^r = r \log_a M$$

(The logarithm of a power of x is the exponent times the logarithms of x.)

Proof:

Let $p = \log_a M$.

? Can you convert the logarithmic equation $p = \log_a M$ to its corresponding exponential equation?

You are correct if you say $a^p = M$.

Now, $(a^p)^r = a^{pr} = M^r$.

Converting back to a logarithmic equation, we obtain: $\log_a M^r = pr$.

But, $p = \log_a M$.

Therefore, $\log_a (M)^r = r \log_a M$.

Example 7

Use laws of logarithms to evaluate the following.

a) $\log_2 \sqrt{8}$ b) $\log_3 81$ c) $\log_{10} \sqrt[3]{0.01}$ d) $\log_4 8 + \log_4 2$ e) $\log_{\frac{1}{2}} \left(\frac{1}{4} \right)$

Solution:

a) $\log_2 \sqrt{8} = \log_2 8^{\frac{1}{2}} = \frac{1}{2} \log_2 2^3 = \frac{1}{2}(3 \log_2 2) = \frac{1}{2} \times 3 \times 1 = \frac{3}{2}$.

b) $\log_3 81 = \log_3 3^4 = 4 \log_3 3 = 4 \times 1 = 4$.

c) $\log_{10} \sqrt[3]{0.01} = \log_{10}(0.01)^{\frac{1}{3}}$

$$= \frac{1}{3} \log_{10} 0.01$$

$$= \frac{1}{3} \log_{10} \frac{1}{100}$$

$$= \frac{1}{3} \log_{10} \frac{1}{10^{-2}}$$

$$= \frac{1}{3} \log_{10} 10^{-2} = \frac{1}{3}(-2\log_{10} 10) = \frac{1}{3} \times (-2) \times 1 = -\frac{2}{3}.$$

d) $\log_4 8 + \log_4 2 = \log_4(8 \times 2) = \log_4 16 = \log_4 4^2 = 2\log_4 4 = 2 \times 1 = 2$

e) $\log_{\frac{1}{2}} \left(\frac{1}{4} \right) = \log_{\frac{1}{2}} \left(\frac{1}{2} \right)^2 = 2 \log_{\frac{1}{2}} \left(\frac{1}{2} \right) = 2 \times 1 = 2$

Theorem 3.3 Logarithms of Quotients

For any positive numbers M, N, and $a > 0$ and $a \neq 1$,
$$\log_a \left(\frac{M}{N} \right) = \log_a M - \log_a N.$$
(The logarithm of a quotient is the logarithm of the numerator minus the logarithm of the denominator.)

Proof: The proof follows from the product law and the power law.

? Do yoo recall the the relation $a^{-n} = \frac{1}{a^n}$? (Definition 3.2)

If your answer is yes then you can see that $N^{-1} = \frac{1}{N}$. We can use this relation in the following step.

$$\log_a\left(\frac{M}{N}\right) = \log_a MN^{-1}$$

$$= \log_a M + \log_a N^{-1} \ . \ . \ . \ \text{using product law}$$

$$= \log_a M + (-1)\log_a N \ . \ . \ . \ \text{using power law}$$

$$= \log_a M - \log_a N.$$

Example 8

Use laws of logarithms to evaluate the following.

   a) $\log_3 54 - \log_3 2$ **b)** $\log_{10} \sqrt{2000} - \log_{10} \sqrt{20}$ **c)** $\log_5 30 + \log_5 15 - \log_5 2$

Solution:

a) $\log_3 54 - \log_3 2 = \log_3 \frac{54}{2} = \log_3 27 = \log_3 3^3 = 3\log_3 3 = 3 \times 1 = 3$

b) $\log_{10} \sqrt{2000} - \log_{10} \sqrt{20} = \log_{10}\left(\frac{\sqrt{2000}}{\sqrt{20}}\right)$

$$= \log_{10}\sqrt{\frac{2000}{20}} = \log_{10}\sqrt{100} = \log_{10} 10 = 1$$

c) $\log_5 30 + \log_5 15 - \log_5 2 = (\log_5 30 + \log_5 15) - \log_5 2$

$$= \log_5(30 \times 15) - \log_5 2$$

$$= \log_5 450 - \log_5 2$$

$$= \log_5\left(\frac{450}{2}\right) = \log_5 225 = \log_5 5^3 = 3\log_5 5 = 3 \times 1 = 3$$

Theorem 3.4 Change of Base

For any positive real number M, $a > 0, b > 0 , a \neq 1$ and $b \neq 1$,
$$\log_a M = \frac{\log_b M}{\log_b a}$$

Proof:

? Do you agree that $a^{\log_a M} = M$? Why?

? What will you get when you change the exponential equation $a^{\log_a M} = M$ to its corresponding logarithmic equation? (In $a^{\log_a M}$, $\log_a M$ is the exponent.)

Let $p = \log_a M$. Then

$a^p = a^{\log_a M} = M$ . . . taking exponent to the base a of both sides.

$\log_b a^p = \log_b M$ . . . taking logarithm to the base b of both sides.

$p \log_b a = \log_b M$ . . . using power law

$$p = \frac{\log_b M}{\log_b a}$$

Therefore, $p = \log_a M = \dfrac{\log_b M}{\log_b a}$.

Example 9

Use laws of logarithms to find:

    a) $\log_{\sqrt{2}} 4$ **b)** $\log_{0.1} 100$

Solution:

    a) $\log_{\sqrt{2}} 4 = \dfrac{\log_2 4}{\log_2 \sqrt{2}}$ by using base change law of logaritms

           $= \dfrac{\log_2 2^2}{\log_2 2^{\frac{1}{2}}}$ because $4 = 2^2$ and $\sqrt{2} = 2^{\frac{1}{2}}$

           $= \dfrac{2 \log_2 2}{\frac{1}{2} \log_2 2}$ by power law of logaritms

           $= \dfrac{2}{\frac{1}{2}}$ because $\log_2 2 = 1$

           $= 2 \times \dfrac{2}{1} = 4$

    b) $\log_{0.1} 100 = \dfrac{\log_{10} 100}{\log_{10} 0.1}$ by using base change law of logaritms

           $= \dfrac{\log_{10} 10^2}{\log_{10} 10^{-1}}$ because $100 = 10^2$ and $0.1 = \dfrac{1}{10} = 10^{-1}$

           $= \dfrac{2 \log_{10} 10}{-1 \log_{10} 10}$ by power law of logaritms

           $= \dfrac{2}{-1}$ because $\log_{10} 10 = 1$

           $= -2$

Logarithms to Base 10 (Common Logarithms)

Dear student, now we are going to discuss about a logarithm when the base is 10. It is called common logarithm. For a number M, $\log_{10} M$ is equal to that power to which 10 must be raised to obtain the number M.

Activity 3.4

Find the value of the following common logarithms.

a) $\log_{10} 10$ **b)** $\log_{10} 1000$ **c)** $\log_{10} 1$ **d)** $\log_{10} 0.1$

Definition 3.7

The logarithm to the base 10 is called common logarithm or decadic logarithm and written as $\log_{10} M$.

☞ A common logarithim is usually written without indicating its base. For example, $log_{10} M$ is simply denoted by $log\, M$

Example 1

Find the value of each of the common logarithms.

    a) $log_{10} \sqrt[3]{10}$ **b)** $log\, 0.0001$ **c)** $log\, 200 - log\, 2$ **d)** $log\left(\frac{0.1}{\sqrt{100}}\right)$

Solution:

    a) $log_{10} \sqrt[3]{10} = log_{10} 10^{\left(\frac{1}{3}\right)}$

$$= \frac{1}{3} log_{10} 10$$

$$= \frac{1}{3} \times 1 = \frac{1}{3}.$$

    b) $log\, 0.0001 = log\left(\frac{1}{10000}\right) = log\left(\frac{1}{10^4}\right)$

$$= log\, 10^{-4}$$

$$= -4\, log\, 10 \quad \text{because } log\, 10 = log_{10} 10 = 1$$

$$= -4.$$

    c) $log\, 200 - log\, 2 = log\left(\frac{200}{2}\right) = log\, 100 = log\, 10^2 = 2\, log\, 10 = 2 \times 1 = 2.$

Dear student, when you tried to compute the values of the common logarithms in activity 3.4 and in the above example you used only properties of logarithm. But common logarithms like $log_{10} 1.23$, $log_{10} 3.57$, $log\, 2.478$ and $log\, 6,920$ can not be easily simplified. To find their values we should use a table called common logarithm table.

Suppose p can be written as $p = m \times 10^c, 1 \leq m < 10$, then the logarithm of p can be read from the common logarithm table (**a table that contains the common logarithm value of a number m such that $1 \leq m < 10$**) which is attached at the last page of this module.

So, $log\, p = log(m \times 10^c) = log\, m + log\, 10^c = log\, m + c.$

That is, $log\, p = log\, m + c.$

The common logarithm of m, $log\, m$ is called the mantissa (fractional part) of the common logarithm of p and c is called the characterstic of the logarithm.

The common logarithm of any two decimal place number between 1.00 and 9.99 can be read directly from the common logarithm table.

Example 2

Using the table of logarithm, calculate

    a) $log\, 1.23$ **b)** $log\, 3.57$ **c)** $log\, 2.478$ **d)** $log\, 6,920$

Solution:

a) We read the number at the intersection of row 1.2 and column 3 to find 0.0899.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
					Logarithm table									Mean Difference					
1.0	0	0.0043	0.0086	0.0128	0.017	0.0212	0.0253	0.0294	0.0334	0.0374	4	8	12	17	21	25	29	33	37
1.1	0.0414	0.0453	0.0492	0.0531	0.0569	0.0607	0.0645	0.0682	0.0719	0.0755	4	8	11	15	19	23	26	30	34
1.2	0.0792	0.0828	0.0864	0.0899	0.0934	0.0969	0.1004	0.1038	0.1072	0.1106	3	7	10	14	17	21	24	28	3
1.3	0.1139	0.1173	0.1206	0.1239	0.1271	0.1303	0.1335	0.1367	0.1399	0.143	3	6	10	13	16	19	23	26	29
1.4	0.1461	0.1492	0.1523	0.1553	0.1584	0.1614	0.1644	0.1673	0.1703	0.1732	3	6	9	12	15	18	21	24	27

So, $\log 1.23 = 0.0899$.

b) Reading the number in row 3.5 under column 7 from the common logarithm table, gives 0.5527.

So, $\log 3.57 = 0.5527$.

c) 2.478 in log 2.478 has three numbers after decimal.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
					Logarithm table									Mean Difference					
2.1	0.3222	0.3243	0.3263	0.3284	0.3304	0.3324	0.3345	0.3365	0.3385	0.3404	2	4	6	8	10	12	14	16	18
2.2	0.3424	0.3444	0.3464	0.3483	0.3502	0.3522	0.3541	0.356	0.3579	0.3598	2	4	6	8	10	12	14	15	17
2.3	0.3617	0.3636	0.3655	0.3674	0.3692	0.3711	0.3729	0.3747	0.3766	0.3784	2	4	6	7	9	11	13	15	17
2.4	0.3802	0.382	0.3838	0.3856	0.3874	0.3892	0.3909	0.3927	0.3945	0.3962	2	4	5	7	9	11	12	14	16
2.5	0.3979	0.3997	0.4014	0.4031	0.4048	0.4065	0.4082	0.4099	0.4116	0.4133	2	3	5	7	9	10	12	14	15

To read the value of $\log 2.478$ from the logarithm table, the steps are as follow:

Step 1. Separate the number 2.478 in log 2.478 as 2.4, 7 and 8.

Step 2. Read the number at the intersection of row 2.4 and column 7, this gives 0.3927.

Step 3. From the mean difference part of the common logarithm table, read the number at the intersection of row 2.4 and column 8, this gives 14. We write this as 0.0014.

Step 4. Add the values obtained in steps 2 and 3 to get $0.3927 + 0.0014 = 0.3941$.

Therefore, $\log 2.478 = 0.3941$.

d) We have, $6{,}920 = 6.92 \times 10^3$.

So, $\log 6{,}920 = \log(6.92 \times 10^3) = \log 6.92 + \log 10^3 = \log 6.92 + 3$.

But, $\log 6.92 = 0.8401$. (reading from the table of common logarithm at the intesection of 6.9 row and column 2).

Therefore, $\log 6{,}920 = \log 6.92 + 3 = 3.8401$.

Example 3

Identify the mantissa and characterstic of each of the common logarithms.

a) $\log 0.00123$ b) $\log 345$ c) $\log 0.01$

Solution:

a) $0.00123 = 1.23 \times 10^{-3}$ and

$\log 0.00123 = \log(1.23 \times 10^{-3}) = \log 1.23 + \log 10^{-3} = \log 1.23 + (-3)$.

The mantissa is $\log 1.23 = 0.0899$ (Refer to the common logarithm table to find $\log 1.23$).

The characterstic is -3.

b) $345 = 3.45 \times 10^2$.

So, the mantissa is $\log 3.45 = 0.5378$ and the characterstic is 2.

c) We have $0.01 = 1.00 \times 10^{-2}$.

Therefore, the mantisa is $\log 1.00 = 0$ and the characterstic is -2.

Antilogarithms

(?) Suppose $\log_{10} M = 0.8175$, $\log_{10} K = 2$ and $\log_{10} R = \frac{1}{2}$, then what are the values of M, K and R?

If $\log M = N$, then M is the antilogarithm (antilog) of N and write $\text{antilog}(N) = M$. When you are asked to find the antilog of a given number N you will try to find a number M such that $\log M = N$. We can use the antilogarithm table attached at the end of this module to read the antilog of a number.

(?) Is $\text{antilog}(\log M) = M$? Why?

For example, given $\log M = 0.8175$ to read the value of $M = \text{antilog}(0.8175)$ from the antilogarithm table, we follow the following steps

Step 1. Separate the number 0.8175 as 0.81, 7 and 5

Anti Logarithm table										Mean Difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.79	6.166	6.18	6.194	6.209	6.223	6.237	6.252	6.266	6.281	6.295	1	3	4	6	7	9	10	11	13
0.80	6.31	6.324	6.339	6.353	6.368	6.383	6.397	6.412	6.427	6.442	1	3	4	6	7	9	10	12	13
0.81	6.457	6.471	6.486	6.501	6.516	6.531	6.546	6.561	6.577	6.592	2	3	5	6	8	9	11	12	14
0.82	6.607	6.622	6.637	6.653	6.668	6.683	6.699	6.714	6.73	6.745	2	3	5	6	8	9	11	12	14
0.83	6.761	6.776	6.792	6.808	6.823	6.839	6.855	6.871	6.887	6.902	2	3	5	6	8	9	11	13	14

Step 2. From the antilogarithm part read the number at the intersection of row 0.81 and column 7, this gives 6.561

Step 3. From the mean difference part, read the number at the intersection of row 0.81 and column 5, this gives 8. We write this as 0.008.

Step 4. Add the values obtained in steps 2 and 3 to get

$M = \text{antilog}(0.8175) = 6.561 + 0.008 = 6.569$

Therefore, $\log(6.569) = 0.8175$

Example 1

Find the antilog of the following numbers.

a) 0.9335 **b)** 3.0913 **c)** -2.1202

Solution:

a) To find antilog(0.9335). That is to find a number M such that $\log M = 0.9335$.

> **Step 1**. Separate the number 0.9335 as 0.93, 3 and 5
>
> **Step 2**. From the antilogarithm part read the number at the intersection of row 0.93 and column 3, this gives 8.570
>
> **Step 3**. From the mean difference part, read the number at the intersection of row 0.93 and column 5, this gives 10. We write this as 0.010.
>
> **Step 4**. Add the values obtained in steps 2 and 3 to get $8.570 + 0.010 = 8.580$
>
> Therefore, antilog$(0.9335) = 8.580$

b) In 3.0913, the characterstic is 3. Therefore aftre finding the antilog of 0.0913, we multiply it by 10^3. To find antilog(0.0913),

> **Step 1**. Separate the number 0.0913 into 0.09, 1 and 3
>
> **Step 2**. From the antilogarithm part read the number at the intersection of row 0.09 and column 1, this gives 1.233.
>
> **Step 3**. From the mean difference part, read the number at the intersection of row 0.09 and column 3, this gives 1. We write this as 0.001.
>
> **Step 4**. Add the values obtained in steps 2 and 3 to get $1.233 + 0.001 = 1.234$
>
> **Step 5.** $1.234 \times 10^3 = 1234.00$. Therefore the antilog$(3.0913) = 1234.00$

c) To find the antilog of negative numbers like -2.1202, we rewrite it as

$$-2.1202 = -2 - 0.1202 + (1 - 1) = (-2 - 1) + (1 - 0.1202)$$
$$= -3 + 0.8798.$$

Then after finding the antilog of 0.8798, we multiply it by 10^{-3} to get the antilog(-2.1202)

Now, to find antilog(0.8798)

> **Step 1**. Separate the number 0.8798 into 0.87, 9 and 8.
>
> **Step 2**. From the antilogarithm part read the number at the intersection of row 0.87 and column 9 , this gives 7.568.
>
> **Step 3**. From the mean difference part, read the number at the intersection of row 0.87 and column 8, this gives 14. We write this as 0.014.
>
> **Step 4**. Add the values obtained in steps 2 and 3 to get $7.568 + 0.014 = 7.582$
>
> **Step 5.** $7.582 \times 10^{-3} = 0.007582$. Therefore the antilog$(-2.1202) = 0.007582$

Computation with Logarithms

Dear students, now you will see how logarithms are used for computations of numbers like $\frac{267 \times 3252}{403}$, $\sqrt{254}$, etc. Specially common logarithms are used in mathematical computations.

In order to compute a given number M, you can perform the following steps:

Step 1. Find $\log M$, using the laws of logarithms.

Step 2. Find the antilogarithm of $\log M$.

Example 2

Approximate the values of the following using logarithm.

a) $\dfrac{267 \times 3252}{403}$ **b)** $\sqrt{254}$

Solution:

a) Let $M = \dfrac{267 \times 3252}{403}$

$$\log M = \log\left(\frac{267 \times 3252}{403}\right) = \log 267 + \log 3252 - \log 403$$

$$= (2 + \log 2.67) + (3 + \log 3.252) - (2 + \log 4.03)$$

$$= 3 + \log 2.67 + \log 3.252 - \log 4.03$$

By reading the values of $\log 2.67$, $\log 3.252$ and $\log 4.03$ from the common logarithm table you can find their values as indited in the step below.

$$= 3 + 0.4265 + 0.5122 - 0.6053$$

$$= 3.3334$$

$\log M = 3.3334$

$M = \text{antilog}(3.3334) = 2155$ (Use the antilog table to find the value of $\text{antilog}(3.3334)$).

b) Let $M = \sqrt{254}$

$\log M = \log(\sqrt{254}) = \frac{1}{2}\log 254 = \frac{1}{2}(2 + \log 2.54) = 1 + \frac{1}{2}\log 2.54$

$= 1 + \frac{1}{2}(0.4048)$

$\log M = 1.2024$

$M = \text{antilog}(1.2024) = 15.93$

Checklist

Dear learner, now it is time to check your understanding about logarithm. Read each question and put a mark (" √ ") in the box that is the appropriate choice to you.

	Yes	No
• Can you explain what is meant by logarithmic expression?	☐	☐

- Do you know the values of $\log_a a$ and $\log_a 1$ when $a > 0$ and $a \neq 1$? ☐ ☐
- Can you state the product rule of logarithm? ☐ ☐
- Can you state power rule of logarithm? ☐ ☐
- Can you state the quotient rule of logarithm? ☐ ☐
- Can you state the rule of change base of logarithm? ☐ ☐
- Can you define common logarithm and antilogarithm? ☐ ☐
- Can you use table for finding logarithm of a given positive number and antilogarithm of a number? ☐ ☐
- Can you define antilogarithm? ☐ ☐

If your response to the above check list questions is positive, proceed to do the self-test exercise given below and check your answer at the end of this module. If not please review the section again and again until you think you have clear understanding about it.

� Self-Test Exercise 2

1. Write the equivalent logarithmic statement for the following equations.

   a) $2^8 = 256$
   b) $2^{-7} = \frac{1}{128}$
   c) $\sqrt[4]{625} = 5$
   d) $81^{-\frac{1}{2}} = \frac{1}{9}$

2. Write the equivalent exponential statement for the following equations.

   a) $\log_{10} 10000 = 4$
   b) $\log_8 8 = 1$
   c) $\log_{10} 0.0001 = -4$
   d) $\log_3 \frac{1}{81} = -4$

3. Find the values of following logarithms.

   a) $\log_9 81$
   b) $\log_8 32$
   c) $\log_{1000} 0.0001$
   d) $\log_{\sqrt{8}} 2$

4. Using properties of logarithms give answer for the following questions.

   a) Find x such that $\log_5 x = \log_5 3$.
   b) Simplify $\log_6 36^p$.
   c) Find x such that $\log_{121} 121 = x$.
   d) Simplify $3^{\log_3 2x}$.

5. Use laws of logarithm to find the values of

   a) $\log_5 \sqrt[3]{5}$
   b) $\log_7 49$
   c) $\log_{10}\left(\frac{1}{10000}\right)$

   d) $\log_{\left(\frac{1}{0.1}\right)}(100)$
   e) $\log_2 \sqrt[3]{\frac{1}{32}}$
   f) $\log_5 100 + \log_5 0.25$

6. Use laws of logarithms to find the values of

   a) $\log_3 162 - \log_3 2$
   b) $\log_2 3 - \log_2 24$
   c) $\log_3 \sqrt{486} - \log_3 \sqrt{2}$
   d) $\log_6 9 + \log_6 120 - \log_6 5$
   e) $\log_5 3 - \log_5 12 + \log_5 20$
   f) $\log_{20} 120 - 2\log_{20} 3 + \log_{20} 30$

7. Use the law $\log_a x = \frac{\log_b x}{\log_b a}$ to find the value of the following expressions.

   a) $\log_{\sqrt{5}} 125$ **b)** $\log_{\sqrt{7}} 49$ **c)** $\log_{\left(\frac{1}{4}\right)} 256$

   d) $\log_{36}\left(\frac{1}{6}\right)$ **e)** $\log_{0.001} 10$

8. Find the values of the following common logarithms.

   a) $\log\left(\frac{\sqrt{1000}}{0.0001}\right)$ **b)** $\log 100\sqrt{0.01}$ **c)** $\log \sqrt[4]{1000}$

9. Using the table of logarithm find the values of the following.

   a) $\log 3.52$ **b)** $\log 1.98$ **c)** $\log 5.4$ **d)** $\log 7.254$

   e) $\log 0.25$ **f)** $\log 8.99$ **g)** $\log 0.0007$ **h)** $\log 500$

10. Identify the characterstic and mantissa of the logarithm of each of the following.

   a) 0.00072 **b)** 0.46 **c)** 603 **d)** $\frac{1}{4}$

   e) 5.5 **f)** 7 **g)** 5420 **h)** 65.615

11. Find

   a) antilog 0.6321 **b)** antilog 0.8440 **c)** antilog 0.7779

   d) antilog 0.8 **e)** antilog 1.4050 **f)** antilog 0.8574

   g) antilog 6.645 **h)** antilog (-0.3)

12. Compute using logarithms.

   a) 3.34×7.45 **b)** $\sqrt[4]{20}$ **c)** $4^{1.53}$

   d) $(5.4)^{2.3} \times (0.33)^{5.2}$ **e)** $\frac{\sqrt{299}}{(3.24)^2}$

<table>
<tr><td>**Section 3**</td><td>**The Exponential Functions and Their Graphs**</td></tr>
</table>

Dear learner, in this section, we define exponential functions. We will sketch their graphs and see properties of the graphs of the functions.

Section Objectives

By the end of this section, you will be able to: -

- define an exponential function.
- draw the graph of a given exponential function.
- describe the graphical relationship of exponential functions having bases reciprocal to each other.
- explain the properties of an exponential function by using its graph.
- define natural exponential function.

3.1 Exponential Functions

Dear learner, activity 3.5 shows how an exponential function is formed. Fill the table carefully and see the relation between the time t and the number of bacteria. The solution of problem c in activity 3.5 is an exponential function.

Activity 3.5

Consider a single bacterium which divides every hour.

a) Find the number of bacteria after one hour, two hours, three hours, four hours, and t hours.

b) Complete the following table.

Time in hour (t)	0	1	2	3	4	…	t
Number of bacteria	1						

c) Write a formula to calculate the number of bacteria after t hours

Definition 3.8

The exponential function f with base a is denoted by $f(x) = a^x$, where $a > 0$, $a \neq 1$ and x is any real number.

? If the value of a in definition 3.8 is 1, then what will be the value of a^x? Do you remember the name of the function $f(x) = a^x$ when a is 1?

Example 1

Given $f(x) = 3^x$. Evaluate the following.

a) $f(2)$
b) $f(0)$
c) $f(-1)$

Solution:

a) $f(2) = 3^2 = 9$.
b) $f(0) = 3^0 = 1$.
c) $f(-1) = 3^{-1} = \frac{1}{3}$.

Example 2

Write each of the following functions in the form $f(x) = 2^{kx}$ or $f(x) = 3^{kx}$ for a suitable constant k.

a) $f(x) = 4^x$
b) $f(x) = \sqrt{2}^x$
c) $f(x) = \left(\frac{1}{9}\right)^x$

Solution:

a) $f(x) = 4^x = (2^2)^x = 2^{2x}$

b) $f(x) = \sqrt{2}^x = \left(2^{\frac{1}{2}}\right)^x = 2^{\frac{1}{2}x}$

c) $f(x) = \left(\frac{1}{9}\right)^x = \left(\frac{1}{3^2}\right)^x = (3^{-2})^x = 3^{-2x}$

3.2 Graphs of Exponential Functions

In this subsection, we will discuss how to draw graphs of exponential function. The graphs can be drawn by preparing table of values, plotting the points (x, y) from the table on the xy-plane and after these we join them by a smooth curve. **A smooth curve has no sharp corner.**

Example 1

Draw the graph of the exponential function $f(x) = 2^x$.

Solution:

First, we calculate values of $f(x)$ for some integer values of x and prepare a table of these values.

$f(-3) = \frac{1}{8}, f(-2) = \frac{1}{4}, f(-1) = \frac{1}{2}, f(0) = 1, f(1) = 2, f(2) = 4, f(3) = 8$

(?) Do you think $f(x) = 2^x$ is defined for all real number x? What do you think about the domain of f?

Next, we prepare table of values as blow.

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Then we plot the points on the xy-plane and join them by a smooth curve as shown in the figure 3.1.
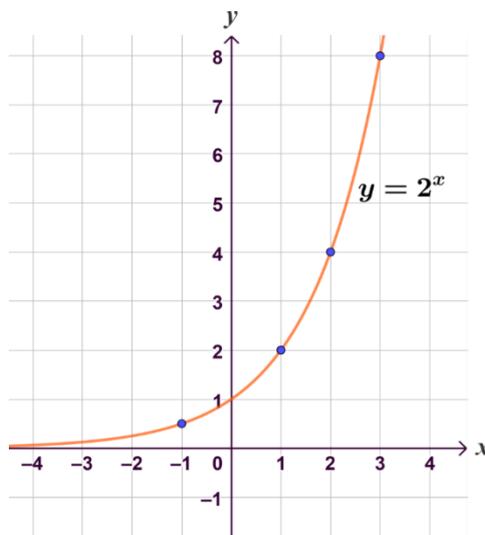
Figure 3.1: Graph of $f(x) = 2^x$

? What is the y-intercept of the function? Do you think the function has an x- intercept? Why? What do you think about the range of f?

From the graph we see that **the domain of $f(x) = 2^x$ is the set of all real numbers and its range is the set of positive real numbers.**

By looking at figure 3.1, you can attempt activity 3.6. This gives you additional properties of the graph of f.

Activity 3.6

1. For which values of x $f(x) = 2^x > 1$?
2. For which values of x $f(x) = 2^x < 1$?
3. Does $f(x) = 2^x$ increase as x increases?
4. What happens to the graph of $f(x) = 2^x$ as x gets larger and larger without bound?
5. What happens to the graph of $f(x) = 2^x$ when x is negative and $|x|$ is very large?
6. Is there a line that the graph of $f(x) = 2^x$ approaches but never touches when x is negative and $|x|$ is very large? What is that line?

Example 2

Draw the graph of the exponential function $f(x) = \left(\frac{1}{2}\right)^x$.

Solution:

First, we calculate values of $f(x)$ for some integer x and prepare a table of these values.

$f(-3) = 8, f(-2) = 4, f(-1) = 2, f(0) = 1, f(1) = \frac{1}{2}, f(2) = \frac{1}{4}, f(3) = \frac{1}{8}$

? Do you think $f(x) = 2^x$ is defined for all real number x? What do you think about the domain of f?

Next, we prepare table of values as blow.

x	-3	-2	-1	0	1	2	3
$f(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Then we plot the points on the xy-plane and join them by a smooth curve as shown in the figure 3.2.
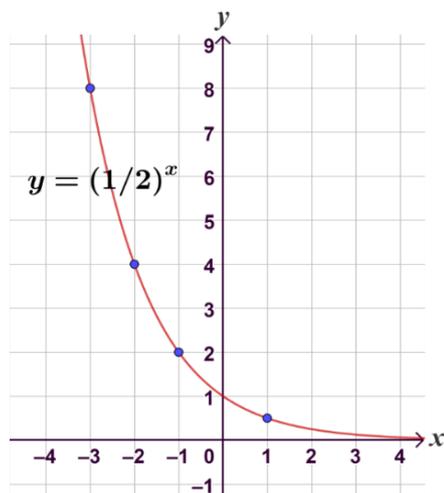
Figure 3.2: Graph of $f(x) = \left(\frac{1}{2}\right)^x$

? What is the y-intercept of the function? Do you think the function has an x- intercept? Why? What do you think about the range of f?

Dear student, the graph of the exponential function $f(x) = a^x$, $a > 0$ and $a \neq 1$ has one of the following shapes. By looking at these graphs you can observe the basic characterstics of the exponential function which are summarized below.

(a)   (b)
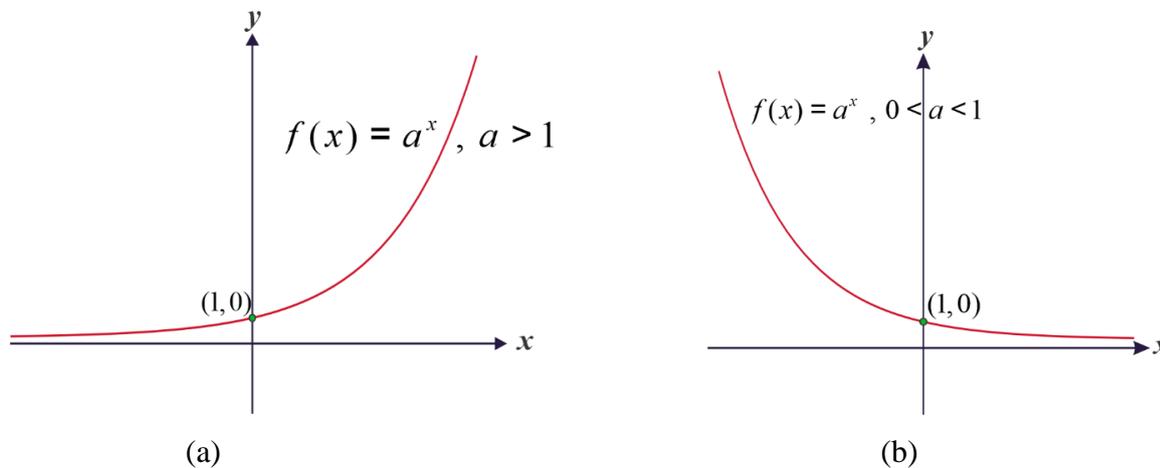
Figure 3.3

Characterstics of Graph of $f(x) = a^x$, $a > 1$

a) Domain: $\mathbb{R}$ = The set of all real numbers.

b) Range: $\mathbb{R}^+$ = The set of all positive real numbers.

c) y-intercept: The point $(0, 1)$.

d) Has no x-intercept.

e) It is increasing function. The value of f increases whenever the value of x increases.

f) The graph goes upward without bound as x gets larger and positive.

g) The graph gets closer to the negative x-axis when x is negative and $|x|$ is large.

Characterstics of Graph of $f(x) = a^x$, $0 < a < 1$.

a) Domain: $\mathbb{R}$ = The set of all real numbers.

b) Range: $\mathbb{R}^+$ =The set of all positive real numbers.

c) y-intercept: The point $(0, 1)$.

d) Has no x intercept.

e) It is decreasing function. The value of f decreases whenever the value of x increases.

f) The graph goes upward without bound when x is negative and $|x|$ is large

g) The graph gets closer to the positive x-axis when x gets larger and positive.

3.3 The Natural Exponential Function

Any positive number can be used as the base for an exponential function, but for the bases, the number denoted by the letter e and 10 are used more frequently. The number e is the most important base and convenient for certain applications.

The number e is defined as the value that $\left(1 + \frac{1}{n}\right)^n$ approaches as n becomes large. The table below shows the values of the expression $\left(1 + \frac{1}{n}\right)^n$ for increasingly large values of n. It appears that $e \cong 2.71828$ correct to five decimal places.

n	$\left(1 + \dfrac{1}{n}\right)^n$
1	2.00000
2	2.25000
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

The approximate value to 20 decimal places is $e \cong 2.71828182845904523536$.

> **Definition 3.9**
>
> ☞ The natural exponential function is the exponential function $f(x) = e^x$ with base e.
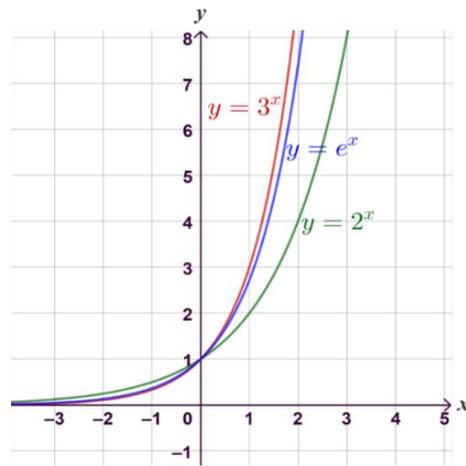
Figure 3.4

Since $2 < e < 3$, the graph of the natural exponential function lies between the graphs of $g(x) = 2^x$ and $h(x) = 3^x$ as shown in the figure 3.4.

Example 1

Use a scientific calculator to evaluate each expression correct to five decimal places.

    a) e^2 **b)** e^{-1} **c)** $e^{3.5}$

Solution:

We use the $[e]$ key on a scientific calculator to evaluate the exponential expressions.

    a) $e^2 \cong 7.38906$.

    b) $e^{-1} \cong 0.36788$.

    c) $e^{3.5} \cong 33.11545$.

Example 2

Construct table of values for some integer values of x, sketch the graphs, find the x-intercept and y-intercept, find the asymptote and give the domain and the range of $y = -e^x$.

x	-2	-1	0	1	2
$y = -e^x$	-0.14	-0.37	-1	-2.72	-7.39

No x-intercept.

The y-intercept is (0, -1).

The asymptote is the line $y = 0$ (x-axis).

The domain is the set of all real numbers.

The range is the set of negative real numbers.
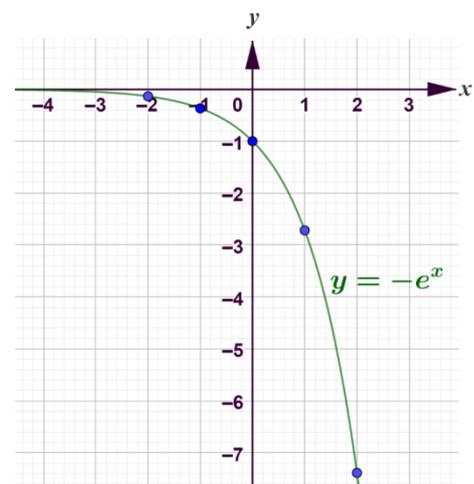
Figure 3.5

Checklist

Dear learner, now it is time to check your understanding about exponential function. Read each question and put a mark (" ✓ ") in the box that is the appropriate choice to you.

	Yes	No
• Can you define exponential function?	☐	☐
• Can you sketch graphs of the exponential functions of the form $y = a^x$ for $a > 0$ and $a \neq 1$?	☐	☐
• Can you describe the properties of the graphs of $y = a^x$ when $a > 1$?	☐	☐
• Can you describe the properties of the graphs of $y = a^x$ when $0 < a < 1$?	☐	☐
• Can you define the natural exponential function?	☐	☐

If your response to the above check list questions is positive, proceed to do the self-test exercise given below and check your answer at the end of this module. If not please review the section again and again until you think you have clear understanding about it.

Self-Test Exercise 3

1. Given $f(x) = \left(\frac{1}{8}\right)^x$. Find the values of

   a) $f(0)$ b) $f(-1)$ c) $f\left(\frac{1}{3}\right)$ d) $f\left(-\frac{1}{3}\right)$

2. Write each of the following functions in the form $f(x) = 5^{kx}$ for a suitable constant k.

   a) $f(x) = 25^x$ b) $f(x) = \sqrt{125}^x$ c) $f(x) = \left(\frac{1}{5}\right)^x$

   d) $f(x) = \sqrt[5]{625}^x$ e) $f(x) = \left(\frac{1}{125}\right)^{\frac{-x}{3}}$

3. For the function $f(x) = 5^x$,

   a) Complete the table of values below.

x	-1	0	1
$y = f(x)$			

   b) Find the intercepts.

   c) Sketch the graph of f, first by plotting the points (x, y) and then joining them by a

smooth curve.

    d) Find the domain and range of f.

4. For the function $f(x) = \left(\frac{1}{5}\right)^x$,

    a) Complete the table of values below.

x	-1	0	1
$y = f(x)$			

    b) Find the intercepts.

    c) Sketch the graph of f, first by plotting the points (x, y) and then joining them by a smooth curve.

    d) Find the domain and range of f.

5. Using the same coordinate system, draw the graphs of $f(x) = 5^x$ and $g(x) = \left(\frac{1}{5}\right)^x$.

Section 4 | The Logarithmic Functions and Their Graphs

4.1 The Logarithmic Functions

✹ Section Overview

In this section, we define logarithmic functions. we will also sketch their graphs and see properties of the graphs of the functions.

> **Definition 3.9**
>
> Let $a > 0$ and $a \neq 1$. The logarithmic function with base a denoted by $y = \log_a x$ is defined by
>
> $$y = \log_a x \text{ if and only if } x = a^y.$$

When evaluating logarithms, remember that $\log_a x$ is the exponent to which a must be raised to obtain x.

Example 1

Use the definition of logarithmic function to evaluate each logarithm function at the indicated value of x.

    a) $f(x) = \log_2 x$, $x = 4$ **b)** $f(x) = \log_4 x$, $x = 1$

    c) $f(x) = \log_3 x$, $x = 27$ **d)** $f(x) = \log_{10} x$, $x = \frac{1}{10}$

Solution:

    a) $f(4) = \log_2 4$. Let $y = \log_2 4$ and by definition 3.9 $y = \log_2 4$ if and only if $2^y = 4$.

But $2^y = 4$ if and only if $y = 2$. Therefore, $\log_2 4 = 2$.

Or $f(4) = \log_2 4 = \log_2 2^2 = 2\log_2 2 = 2 \times 1 = 2$

b) $f(1) = \log_4 1$. Let $y = \log_4 0$ and by definition 3.9 $y = \log_4 0$ if and only if $4^y = 0$.

But $4^y = 0$ if and only if $y = 0$. Therefore, $\log_4 1 = 0$

c) $f(27) = \log_3 27 = \log_3 3^3 = 3\log_3 3 = 3 \times 1 = 3$.

d) $f\left(\frac{1}{10}\right) = \log_{10}\left(\frac{1}{10}\right) = \log_{10} 10^{-1} = -\log_{10} 10 = -1$, b

Example 2

Write each of the following functions in the form $f(x) = k\log_2 x$ or $f(x) = k\log_3 x$ for a suitable constant k.

a) $f(x) = \log_4 x$

b) $f(x) = \log_9 x$

c) $f(x) = \log_{\left(\frac{1}{4}\right)} x$

d) $f(x) = \log_{\sqrt{3}} x$

Solution:

? Can you recall theorem 3.4 (change base rule of logarithm)?

This rule and other rules are applied to find the solution of this example.

a) $f(x) = \log_4 x = \dfrac{\log_2 x}{\log_2 4} = \dfrac{\log_2 x}{\log_2 2^2} = \dfrac{\log_2 x}{2\log_2 2} = \dfrac{\log_2 x}{2} = \dfrac{1}{2}\log_2 x$

b) $f(x) = \log_9 x = \dfrac{\log_3 x}{\log_3 9} = \dfrac{\log_3 x}{\log_3 3^2} = \dfrac{\log_3 x}{2\log_3 3} = \dfrac{\log_3 x}{2} = \dfrac{1}{2}\log_3 x$

c) $f(x) = \log_{\left(\frac{1}{4}\right)} x = \dfrac{\log_2 x}{\log_2 2^{-2}} = \dfrac{\log_2 x}{-2\log_2 2} = -\dfrac{1}{2}\log_2 x$

d) $f(x) = \log_{\sqrt{3}} x = \dfrac{\log_3 x}{\log_3 \sqrt{3}} = \dfrac{\log_3 x}{\frac{1}{2}\log_3 3} = 2\log_3 x$

Example 3

Write each of the following functions in the form $f(x) = c\log_2 kx$ or $f(x) = c\log_5 kx$ for suitable constants c and k.

a. $f(x) = \log_{\left(\frac{1}{8}\right)}\left(\frac{-x}{2}\right)$

b. $f(x) = \log_{\left(\frac{1}{25}\right)}\left(\frac{x}{5}\right)$

Solution:

a. $f(x) = \log_{\left(\frac{1}{8}\right)}\left(\frac{-x}{2}\right) = \dfrac{\log_2\left(\frac{-x}{2}\right)}{\log_2\left(\frac{1}{8}\right)} = \dfrac{\log_2\left(\frac{-x}{2}\right)}{\log_2(2)^{-3}} = \dfrac{\log_2\left(\frac{-x}{2}\right)}{-3\log_2 2} = \dfrac{\log_2\left(-\frac{x}{2}\right)}{-3} = -\dfrac{1}{3}\log_2\left(-\dfrac{1}{2}x\right)$

b. $f(x) = \log_{\left(\frac{1}{25}\right)}\left(\frac{x}{5}\right) = \dfrac{\log_5\left(\frac{x}{5}\right)}{\log_5\left(\frac{1}{25}\right)} = \dfrac{\log_5\left(\frac{x}{5}\right)}{\log_5(5)^{-2}} = \dfrac{\log_5\left(\frac{x}{5}\right)}{-2\log_5 5} = \dfrac{\log_5\left(\frac{x}{5}\right)}{-2} = -\dfrac{1}{2}\log_5\left(\dfrac{1}{5}x\right)$

4.2 Graphs of Logarithmic Functions

Dear learner, in this subsection, we will discuss how to draw graphs of exponential function. The graphs can be drawn by preparing table of values, plotting the points (x, y) from the table on the xy-plane and after these we join them by a smooth curve. **A smooth curve has no sharp corner.**

? Can you find the value of $y = \log_2 x$ for $x = -8, -4, 0$? Why?

What do you think about the domain of the logarithmic function $f(x) = \log_2 x$?

Example 1

Draw the graph of the logarithmic function $f(x) = \log_2 x$.

Solution:

First, we calculate values of $f(x)$ for some values of x which are powers of 2 so that we can find their logarithms and prepare a table of these values.

$f(8) = \log_2 8 = 3,$ $f(4) = \log_2 4 = 2,$ $f(2) = \log_2 2 = 1,$ $f(1) = \log_2 1 = 0,$

$f\left(\frac{1}{2}\right) = \log_2 \frac{1}{2} = -1,$ $f\left(\frac{1}{4}\right) = \log_2 \frac{1}{4} = -2,$ $f\left(\frac{1}{8}\right) = \log_2 \frac{1}{8} = -3$

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$f(x) = \log_2 x$	-3	-2	-1	0	1	2	3

Then we plot the points on the xy-plane and join them by a smooth curve as shown by figure 3.6
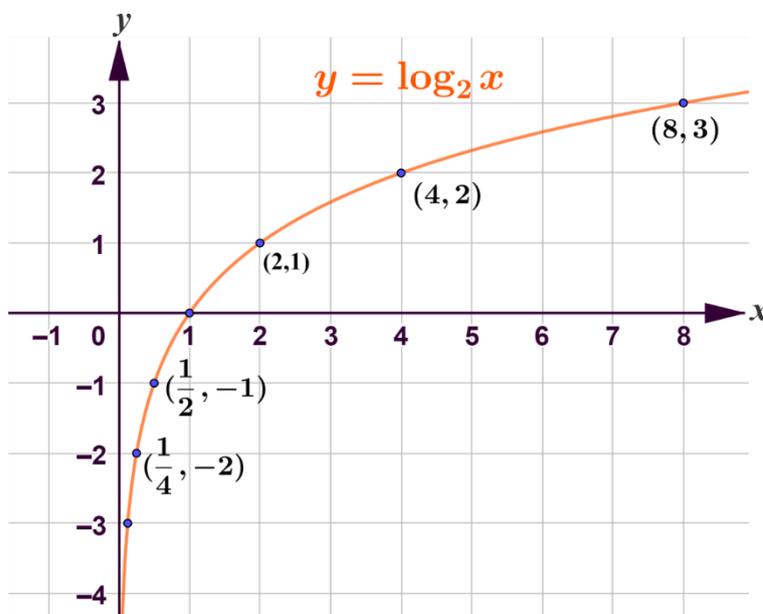
Figure 3.6: Graph of $f(x) = \log_2 x$.

The graph of $f(x) = \log_a x$, $a > 1$ and its basic characteristics are listed below for you.
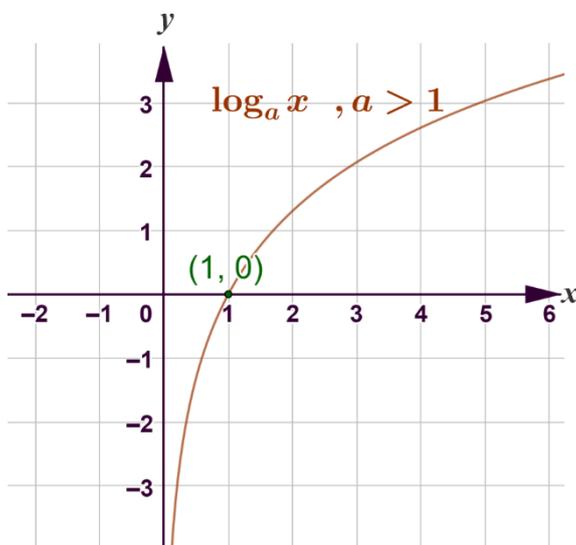
Figure 3.7: Graph of $f(x) = \log_a x$, $a > 1$

1. Domain: $\mathbb{R}^+$ =The set of all positive real numbers.

2. Range: $\mathbb{R}$ = The set of all real numbers.

3. x-intercept: $(1, 0)$

4. It has no y intercept. It does not intersect the y-axis.

5. It is increasing function. The value of f increases whenever the value of x increases.

6. The graph goes upward as x gets larger and positive.

7. The graph gets closer to the negative y-axis when x gets closer to 0 from the right.

▊ Example 2

Draw the graph of the logarithmic function $f(x) = \log_{\frac{1}{2}} x$.

Solution:

First, we calculate values of $f(x)$ for some values of x which are powers of $\frac{1}{2}$ so that we can find their logarithms and prepare a table of these values.

$f(8) = \log_{\frac{1}{2}} 8 = -3, \quad f(4) = \log_{\frac{1}{2}} 4 = -2, \quad f(2) = \log_{\frac{1}{2}} 2 = -1, \quad f(1) = \log_{\frac{1}{2}} 1 = 0$

$f\left(\frac{1}{2}\right) = \log_{\frac{1}{2}} \frac{1}{2} = 1, \quad f\left(\frac{1}{4}\right) = \log_{\frac{1}{2}} \frac{1}{4} = 2, \quad f\left(\frac{1}{8}\right) = \log_{\frac{1}{2}} \frac{1}{8} = 3$

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$f(x) = \log_{\frac{1}{2}} x$	3	2	1	0	-1	-2	-3

Then we plot the points on the xy-plane and join them by a smooth curve as shown in figure 3.8.
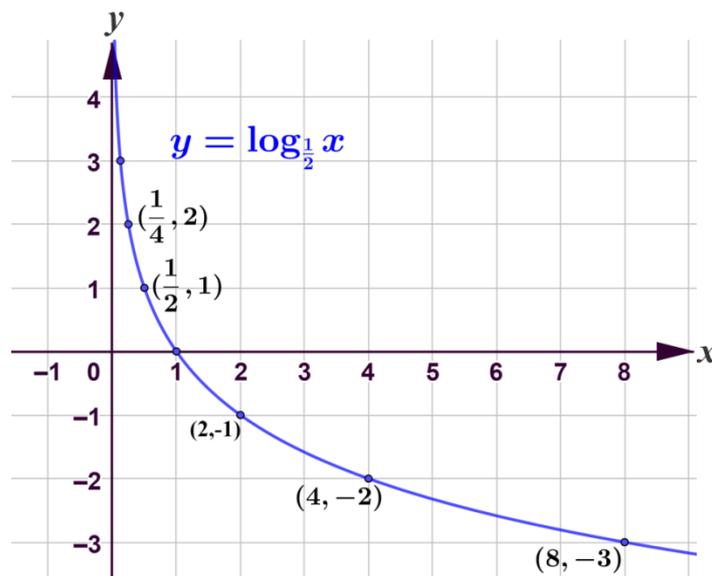
Figure 3.8: Graph of $f(x) = \log_{\frac{1}{2}} x$

The graph of $f(x) = \log_a x$, $0 < a < 1$ and its basic characteristics are listed below for you.
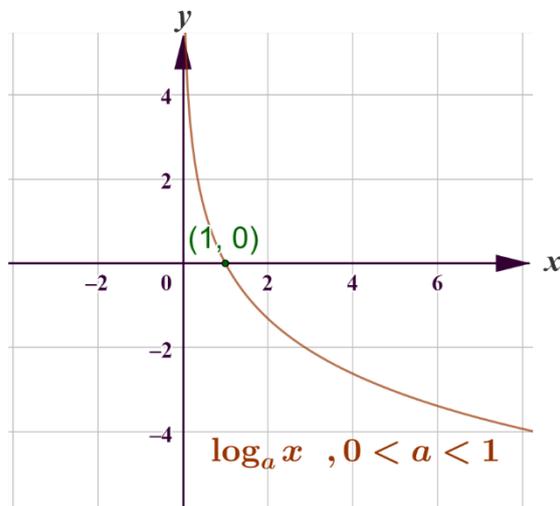
Figure 3.9 Graph of $f(x) = \log_a x$, $0 < a < 1$

1. Domain: $\mathbb{R}^+$ =The set of all positive real numbers.

2. Range: $\mathbb{R}$ = The set of all real numbers.

3. x-intercept: $(1, 0)$

4. It has no y intercept. It does not intersect the y-axis.

5. It is decreasing function. The value of f decreases whenever the value of x increases.

6. The graph goes downward as x gets larger and positive.

7. The graph gets closer to the positive y-axis when x gets closer to 0 from the right.

4.3 Natural Logarithms

Dear student, recall from section 3.3 about the number e. Now, we are going to define this number in relation with lagarithm.

Definition 3.10

The logarithm of a number to the base e is called **natural logarithm** and it is written as

$$\log_e x = \ln x.$$

and $f(x) = log_e x \ or \ f(x) = ln \ x$ is called natural logarithmic function.

Example 1

Find the value of each of the following natural logarithms.

a) $\ln 1$ **b)** $\ln e$ **c)** $\ln e^3$ **d)** $\ln \sqrt[3]{e}$ **e)** $\ln \frac{1}{e}$

Solution:

a) $\ln 1 \ = \ \log_e 1 = 0$

b) $\ln e = \log_e e = 1$

c) $\ln e^3 = 3 \ln e = 3$

d) $\ln \sqrt[3]{e} \ = \ln e^{\frac{1}{3}} = \frac{1}{3}\ln e = \frac{1}{3}$

e) $\ln \frac{1}{e} \ = \ln e^{-1} = -\ln e = -1$

Checklist

Dear learner, now it is time to check your understanding of logarithmic functions and their graphs.

Read each question and put a mark (" $\checkmark$ ") in the box that is the appropriate choice to you.

	Yes	No
• Can you define logarithmic function?	☐	☐
• Can you sketch graphs of logarithmic functions of the form $y = \log_a x$ for $a > 0$ and $a \neq 1$?	☐	☐
• Can you list the properties of the graphs of $y = \log_a x$ when $a > 1$?	☐	☐
• Can you list the properties of the graphs of $y = \log_a x$ when $0 < a < 1$?	☐	☐
• Can you define natural logarithm?	☐	☐

If your response to the above check list questions is positive, proceed to do the self-test exercise given below and check your answer at the end of this module. If not please review the section again and again until you think you have clear understanding about it.

Self-Test Exercise 4

1. Given $f(x) = \log_{\left(\frac{1}{9}\right)} x$. Find the values of

   a. $f(3)$ **b.** $f(-3)$

   c. $f\left(\frac{1}{27}\right)$ **d.** $f(\sqrt{3})$

2. Write each of the following functions in the form $f(x) = k\log_5 x$ for a suitable constant k.

   a. $f(x) = \log_{25} x$ **b.** $f(x) = \log_{\sqrt{125}} x$

   c. $f(x) = \log_{\left(\frac{1}{5}\right)} x$ **d.** $f(x) = \log_{\sqrt[3]{5}} x$

3. For the function $f(x) = \log_5 x$

   a) Complete the table of values below

x	$\dfrac{1}{125}$	$\dfrac{1}{25}$	$\dfrac{1}{5}$	1	5
$y = f(x)$					

   b) Find the intercepts.

   c) Sketch the graph of f, first by plotting the points (x, y) and then joining them by a smooth curve.

   d) Find the domain and range of f.

4. For the function $f(x) = \log_{\left(\frac{1}{5}\right)} x$

   a) Complete the table of values below.

x	$\dfrac{1}{125}$	$\dfrac{1}{25}$	$\dfrac{1}{5}$	1	5
$y = f(x)$					

   b) Find the intercepts.

   c) Sketch the graph of f, first by plotting the points (x, y) and then joining them by a smooth curve.

   d) Find the domain and range of f.

5. Find the values of:

   a) $\ln\sqrt[6]{e}$ **b)** $\ln\left(\frac{1}{e^4}\right)$ **c)** $e^{\ln 7}$ **d)** $\ln\sqrt[5]{e^3}$

   e) $\ln(e^4 \times e^3)$ **f)** $\ln(e^x \cdot e^{-y})$ **g)** $\ln\left(\frac{e^{2x}}{e^{-y}}\right)$

Section 5	Relation Between Exponential and Logarithmic Functions with the Same Base

❈ Section Overview

Dear learner, in this section, we will look at the relation between exponential and logarithmic functions with the same base. Then we will see this relation sketch their graphs on the same xy-plane.

> ### 🎯 Section Objectives
>
> **By the end of this section, you will be able to: -**
>
> - Sketch the graphs of exponential and logarithmic functions with the same base on the same xy-plane.
>
> - relate exponential and logarithmic functions with the same base.

Dear student, consider the tables of values that we have constructed in the previous sections for the exponential function $y = 2^x$ and logarithmic function $y = \log_2 x$. They have the same base 2.

x	-3	-2	-1	0	1	2	3
$y = 2^x$	$\dfrac{1}{8}$	$\dfrac{1}{4}$	$\dfrac{1}{2}$	1	2	4	8

x	$\dfrac{1}{8}$	$\dfrac{1}{4}$	$\dfrac{1}{2}$	1	2	4	8
$y = \log_2 x$	-3	-2	-1	0	1	2	3

We see that the values of x and y are interchanged in the two tables. That is, the domain of $y = 2^x$ is the range of $y = \log_2 x$, the range of $y = 2^x$ is the domain of $y = \log_2 x$ and vice-versa.

You can look back to section 3 and section 4 for the graphs of the two functions and if we try to sketch the graphs of both functions and the graph of the linear function $y = x$ on the same xy-plane then the result will be as shown in the figure 3.10.
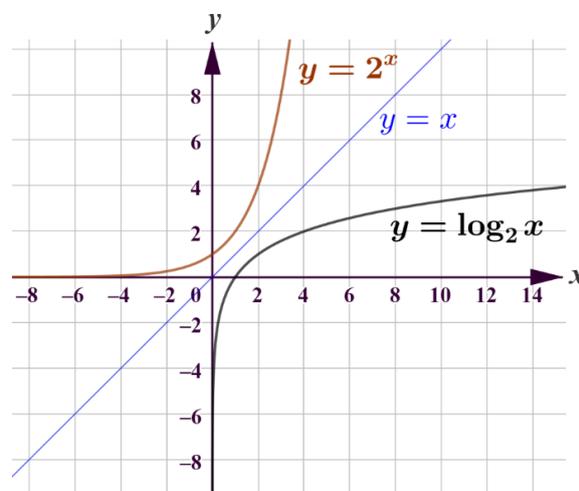
Figure 3.10

If $x = 2$ then the value of $y = 2^x$ is 4. This means (2, 4) is on the graph of $y = 2^x$. If $x = 4$ then the value of $y = \log_2 x$ is 2. This means (4, 2) is on the graph of $y = \log_2 x$.

? Can you locate the position of the points (2, 4) and (4, 2) on figure 3.19?

The line $y = x$ is the perpendicular bisector of the line segment joining the points (2, 4) and (4, 2). In general, for every point (a, b) on the graph of $y = 2^x$ the point (b, a) lies on the graph of $y = \log_2 x$ such that the line $y = x$ is the perpendicular bisector of the line segment joining the two points. Because of this the graph of $y = \log_2 x$ is the reflection of the graph of $y = 2^x$ along the line $y = x$ as shown in figure 3.10.

The relation between the functions $y = a^x$ and $y = \log_a x$, $a > 1$ is shown graphically in figure 3.11.
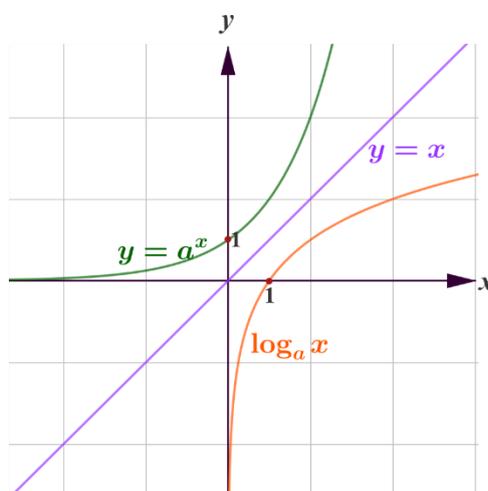
Figure 3.11: Graph of $y = \log_a x$ and $y = a^x$ for $a > 1$

From figure 3.11, observe that:

1. The domain of $y = a^x$ is the set of all real numbers, that is the range of $y = \log_a x$.

2. The range of $y = a^x$ is the set of all positive real numbers, that is the domain of $y = \log_a x$.

   a. Domain of $y = a^x$ = Range of $y = \log_a x$.

   b. Range of $y = a^x$ = Domain of $y = \log_a x$.

Figure 3.12 shows graphs of the family of logarithmic functions with bases 2, 3, 4, 5 and 10. These graphs are the reflections of the graphs of $y = 2^x$, $y = 3^x$, $y = 4^x$, $y = 5^x$ and $y = 10^x$ along the line $y = x$, respectively.
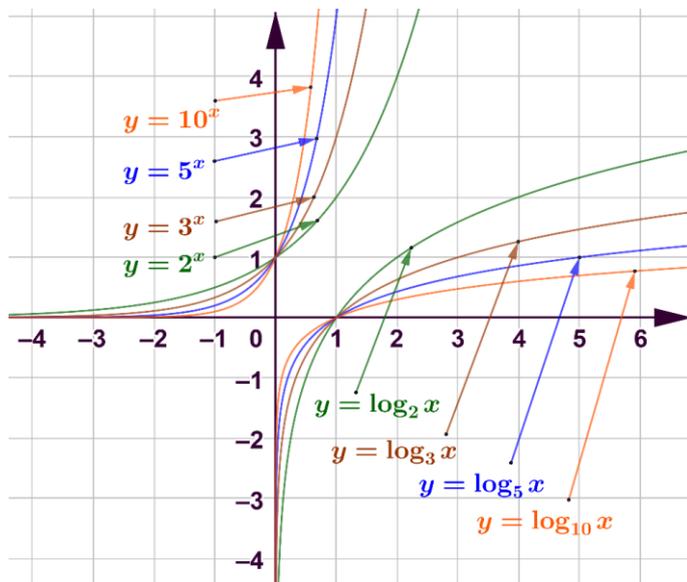
Figure 3.12

Example 1

For the exponential function $y = \left(\frac{1}{2}\right)^x$ and the logarithmic function $y = \log_{\frac{1}{2}} x$,

a. Complete the table of values below.

x	-3	-2	-1	0	1	2	3
$f(x) = \left(\frac{1}{2}\right)^x$							

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$g(x) = \log_{\frac{1}{2}} x$							

b. Sketch their graphs on the same xy-plane.

c. Find the domain and the range of the functions.

d. State the relation that exists between the domain and the range of the functions.

Solution:

a.

x	-3	-2	-1	0	1	2	3
$f(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$g(x) = \log_{\frac{1}{2}} x$	3	2	1	0	-1	-2	-3
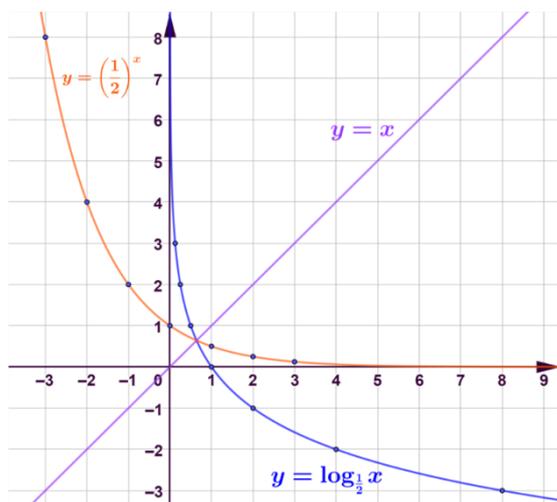
Figure 3.13: Graph of $y = \left(\frac{1}{2}\right)^x$ and $y = \log_{\frac{1}{2}} x$

c. The domain of $f(x)$ is the set of all real numbers and its range is the set of positive real numbers. The domain of $g(x)$ is the set of positive real numbers and its range is the set of all real numbers.

d. Domain of $f(x)$ = Range of $g(x)$ and Range of $f(x)$ = Domain of $g(x)$.

Checklist

Dear learner, now it is time to check your understanding about operations on polynomial functions.

Read each question and put a mark (" ✓ ") in the box that is the appropriate choice to you.

	Yes	No
• Can you draw graphs of exponential and logarithmic functions with the same base on the same xy-plane.	☐	☐
• Can you relate the domain and range of $y = a^x$ with the domain and range of $y = \log_a x$?	☐	☐

If your response to the above check list questions is positive, proceed to do the self-test exercise given below and check your answer at the end of this module. If not please review the section again and again until you think you have clear understanding about it.

Self-Test Exercise 5

Let $h(x) = \left(\frac{1}{5}\right)^x$ and $k(x) = \log_{\frac{1}{5}} x$.

1. Sketch the graphs of $h(x)$ and $k(x)$ on the same xy-plane.
2. Find the domain and the range of $h(x)$.
3. Find the domain and the range of $k(x)$.
4. Compare the domain of $h(x)$ with the range of $k(x)$.

5. Compare the range of $h(x)$ with the domain of $k(x)$.

| Section 6 | Solving Exponential and Logarithmic Equations |

❈ Section Overview

In this section, we will solve exponential and logarithmic equations. We will also look at some procedures which are necessary to solve these types of equations.

Section Objectives

By the end of this section, you will be able to: -

- Solve simple exponential equations by using the properties of Exponent

- Use the properties of logarithm to solve simple logarithmic equation.

6.1 Solving Exponential Equations

> Do you remember the name of the equations $2x + 5 = 2$, $3x^2 + 5x + 7 = 0$ and $2x^3 + x^2 + 4x - 5 = 0$?

An equation in which the variable occurs in the exponent is called an exponential equation. For instance,

$2^x = 16$ and $3^{5x} = 81$ are exponential equations.

To solve exponential equations, we follow the following 3 step procedure.

1. Isolate the exponential expression on one side of the equation.

2. Solve for the variable.

Moreover, we use the following property:

> ☞ **Base-exponent property**
>
> For any real numbers $x, y, \ a > 0, \ a \neq 1, \ a^x = a^y$ if and only if $x = y$

Example 1

Solve $2^{(2x-1)} = 8$.

Solution:

Since $8 = 2^3$, we have

$2^{(2x-1)} = 8 = 2^3$

$\quad 2^{(2x-1)} = 2^3$

$\quad 2x - 1 = 3$ (base exponent property)

$x = 2$ (Solving the linear equation for x)

So, the solution is $x = 2$.

Example 2

Solve $2^{x^2 - 3x} = 16$.

Solution:

Since $16 = 2^4$, we have $2^{x^2 - 3x} = 2^4$

$x^2 - 3x = 4$ (Base exponent property)

$x^2 - 3x - 4 = 0$ (Find two numbers whose sum is -3 and whose product is -4. These numbers are 1 and -4. Then we rewrite the equation like below.)

$x^2 - x - 4x - 4 = 0$.

$x(x - 1) - 4(x - 1)$

$(x + 1)(x - 4) = 0$ (Factorizing $x^2 - 3x - 4$.)

So, the solutions are $x = -1$ and $x = 4$.

(?) Can you recall the general formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve the general quadratic equation $ax^2 + bx + c = 0$? Can you solve $x^2 - 3x - 4 = 0$ using this formula?

6.2 Solving Logarithmic Equations

A logarithmic equation is **an equation that involves the logarithm of an expression containing a variable**. For instance, $\log_2(x + 3) = 4$ is logarithmic equation.

Since the logarithm of non-positive (0 and negative) numbers does not exist, before trying to find the solution of $\log_2(x + 3) = 4$, you have to restrict x such that $x + 3 > 0$. That is, $x > -3$. We accept the solution if the value of x is above -3. The set of all numbers greater than -3 is called the **universal set** or simply the **universe** of the equation $\log_2(x + 3) = 4$.

We use the following property to solve logarithmic equations.

☞ For any positive real numbers $x, y, a > 0$ and $a \neq 1$

$\log_a x = \log_a y$ if and only if $x = y$.

We use the following procedures to solve logarithmic equations.

1. State the universe.
2. Collect the logarithmic term on one side of the equation.
3. Write the equation in exponential form.
4. Solve for the variable.

Example 1

Solve the logarithmic equation $\log_2(x + 3) = 4$.

Solution:

If $x + 3 > 0$ then the universe is $x > -3$.

$$\log_2(x + 3) = 4$$
$$x + 3 = 2^4 \qquad \text{exponential form}$$
$$x + 3 = 16$$
$$x = 13 > -3$$

So, the solution is $x = 13$.

Example 2

Solve $\log_3(5x - 2) = 2 - \log_3 4$.

Solution:

If $5x - 2 > 0$ then $x > \frac{2}{5}$.

$\log_3(5x - 2) = 2 - \log_3 4$

$\log_3(5x - 2) + \log_3 4 = 2$

$\log_3 4(5x - 2) = 2$ (Product law of logarithm)

$\log_3(20x - 8) = 2$

$20x - 8 = 3^2$ (Definition of logarithm)

$20x - 8 = 9$

$x = \frac{17}{20}$

$x = \frac{17}{20} > \frac{2}{5}$

Therefore, $x = \frac{17}{20}$ is the solution.

Checklist

Dear learner, now it is time to check your understanding about operations on polynomial functions.

Read each question and put a mark (" $\checkmark$ ") in the box that is the appropriate choice to you.

	Yes	No
• Can you solve exponential equations using property of exponent?	☐	☐
• Can you solve simple logarithmic equations using the properties of logarithm?	☐	☐

If your response to the above check list questions is positive, proceed to do the self-test exercise given below and check your answer at the end of this module. If not please review the section again and again until you think you have clear understanding about it.

Self-Test Exercise 6

1. Solve for x.

   a) $10^x = 10000$ **b)** $4^{2-3x} = \frac{1}{16}$ **c)** $3^{3x-1} = 9^{2-x}$

   d) $\frac{1}{25} = \left(\frac{1}{125}\right)^x$ **e)** $\left(\frac{1}{6}\right)^{3x+2} \cdot 216^{3x} = \frac{1}{216}$ **f)** $3^{2x-2} + 3^{2x} = 10$

   g) $2^x + 3^{x-2} = 3^x - 2^{x+1}$ **h)** $5^{x(x-3)} = \frac{1}{25}$

2. State the universe and solve for x.

   a) $\log_3(2 - 3x) = 2$ **b)** $\log_{\sqrt{3}} x = 8$

   c) $\log_2(2x^2 - x) = 0$ **d)** $\log_4 x + \log_4(x - 3) = 1$

   e) $\log(4x^2 - 9) - \log(2x + 3) = 1$ **f)** $\log_6(x + 4) + \log_6(x - 2) = \log_6 4x$

   g) $\log(x + 6) - \log x = \log 2$

Section 7   Applications

In this section we will look at two applications of exponential function. These are calculating compound interest and population growth.

Section Objectives

By the end of this section, you will be able to: -

- calculate compound interest.
- compute population growth.

7.1 Compound Interest

Exponential functions occur in evaluating compound interest. Suppose an amount of money P, called the principal, is invested at an annual interest rate r, compounded once a year, then after a year the interest is Pr. If the interest is added to the principal at the end of the year, the new amount $A(1)$ of money is,

$$A(1) = P + Pr = P(1 + r)$$

If the interest is reinvested, then the new principal is $A(1) = P(1 + r)$, and after another year the interest is $rA(1)$, then the amount after the end of the second year, $A(2)$ is

$$A(2) = A(1) + rA(1) = A(1)(1 + r) = P(1 + r)(1 + r) = P(1 + r)^2$$

Similarly, at the end of the third year, the amount $A(3)$ is $A(3) = P(1 + r)^3$.

Generally, after the end of t years, the amount $A(t)$ is $A(t) = P(1 + r)^t$.

Observe that this is an exponential function with base $1 + r$.

Let n be the number of compounding per year and t be the number of years. Then the product nt represents the total number of times the interest will be compounded and the interest rate per compounding period is $\frac{r}{n}$. This leads to the following formula indicated as in theorem 3.5 for the amount after t years.

Theorem 3.5 Compound interest

Compound interest is calculated by the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where $A(t)$= amount after t years,

P = principal,

r = interest rate per year,

n = number of times interest is compounded per year and

t = number of years.

Example 1

A total of birr 100 is invested at an interest rate of 7% per year. Find the amounts in the account after 5 years if the interest is compounded

    a) annually   **b)** semi-annually   **c)** quarterly   **d)** monthly   **e)** daily.

Solution:

    a) Here we have $P = 100$, $r = 7\% = 0.07$, $n = 1$ and $t = 5$.

        Using the formula for compound interest with n compounding per year, we have

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \quad \ldots \text{ Formula for compound interest}$$

$$= 100\left(1 + \frac{0.07}{1}\right)^{1(5)} \ldots \text{ Substitute for } P, r, n \text{ and } t$$

$$= 100(1.07)^5 \quad \ldots \text{ Simplifying}$$

$$\approx 140.26 \quad \ldots \text{ use a calculator}$$

        Therefore, the amount in the account after 5 years will be about Birr 140.26.

    b) For semi-annually compounding, $n = 2$. Hence, after 5 years at 7% rate, the amount in the account is

$$A(5) = 100\left(1 + \frac{0.07}{2}\right)^{2(5)} = 100(1.035)^{10} \approx 141.06 \text{ birr.}$$

c) For quarterly compounding, $n = 4$. Thus, after 5 years at 7% rate, the amount in the account is

$$A(5) = 100\left(1 + \frac{0.07}{4}\right)^{4(5)} = 100(1.0175)^{20} \approx 141.48 \text{ birr.}$$

d) For monthly compounding, $n = 12$. So, after 5 years at 7% rate, the amount in the account is

$$A(5) = 100\left(1 + \frac{0.07}{12}\right)^{12(5)} = 100(1.00583)^{60} \approx 141.76 \text{ birr.}$$

e) For daily compounding, $n = 365$. Therefore, after 5 years at 7% rate, the amount in the account is

$$A(5) = 100\left(1 + \frac{0.07}{365}\right)^{365(5)} = 100(1.00019)^{1825} \approx 141.90 \text{ birr.}$$

Note

The interest paid increases as the number of compounding period n increases.

3.6.2 Population Growth

> The exponential function
>
> $$P(t) = P_0 e^{kt}, k > 0$$
>
> is a mathematical model of many kinds of population growth. In this function, P_0 is the population at initial time t_0, $P(t)$ is the population after time t, and k is called the exponential growth rate.

Example 2

In 2013 E.C, the population of a country was about 110 million, and the exponential growth rate was 2.3 % per year.

a) Find the exponential growth function.

b) Estimate the population in 2018 E.C.

c) How many years will it take for the population to be doubled?

Solution:

a) Here $P_0 = 110$ million, the population in the year 2013 $(t = 0)$ and the growth rate $k = 2.3\% = 0.023$. So, the exponential growth function is:

$$P(t) = (110,000,000)e^{0.023t}.$$

b) In the year 2018, $t = 5$. To find the population in 2018, we substitute 5 for t, i.e.

$$P(5) = (110,000,000)e^{0.023(5)} = (110,000,000)e^{0.115} \approx 123,406,078.$$

Therefore, the population will be about 123,406,078 in 2018.

c) We find t for which $P(t) = 2P_0 = 2(110,000,000) = 220,000,000$.

To find the time, we solve the equation:

$$P(t) = (110,000,000)e^{0.023t}$$

$$220,000,000 = (110,000,000)e^{0.023t}$$

$$2 = e^{0.023t} \quad \ldots \quad \text{dividing each side by } 100,000,000$$

$$\ln 2 = \ln e^{0.023t} \quad \ldots \quad \text{taking the natural logarithm of both sides}$$

$$\ln 2 = 0.023t \underbrace{\ln e}_{=1} \quad \ldots \quad \text{power law of logarithm}$$

$$\ln 2 = 0.023t$$

$$t = \frac{\ln 2}{0.023} \quad \ldots \quad \text{dividing each side by } 0.023$$

$$t \approx \frac{0.69315}{0.023} \quad \text{because } \ln 2 \approx 0.69315$$

$$t \approx 30.14$$

Hence, it takes about 30.14 years for the population of the country to be double.

3.6.3 The pH Scale

Chemists measured the acidity of a solution by giving its hydrogen ion concentration until Soren Peter Lauritz Sorensen, in 1909, proposed a more convenient measure. He defined

$$pH = -\log[H^+]$$

where $[H^+]$ is the concentration of hydrogen ions measured in moles per liter (M).

Solutions with a $pH = 7$ are defined **neutral**, those with $pH < 7$ are **acidic**, and those with $pH > 7$ are **basic**.

Example 3

The hydrogen ion concentration of a sample of human blood was measured to be $[H^+] = 4.53 \times 10^{-8}$ M. Find the pH and determine whether the blood is acidic or basic.

Solution:

$$\text{We have} \quad pH = -\log[H^+] = -\log[4.53 \times 10^{-8}]$$

$$= -[\log(4.53) + \log 10^{-8}] \quad \ldots \quad \text{product law of logarithm}$$

$$= -[0.6561 - 8] \quad (\text{from log table } \log(4.53) = 0.6561)$$

$$= 7.344$$

Since $pH = 7.344 > 7$, the blood is basic.

Example 4

The most acidic rainfall ever measured occurred in Scotland in 1974 and its pH was 2.4. Find the hydrogen ion concentration.

Solution:

$$pH = -\log[H^+]$$

$$2.4 = -\log[H^+] \quad \log[H^+] = -2.4 \text{ ... multiply both sides by -1.}$$

$$\log[H^+] = (3 - 2.4) - 3 \quad (3 - 3 = 0 \text{ ading 0 to a number make no change})$$

$$\log[H^+] = 0.6 + (-3)$$

$$\text{antilog}(\log[H^+]) = \text{antilog}(0.6 + (-3))$$

$$[H^+] = 3.981 \times 10^{-3}$$

So, the hydrogen ion concentration of the rainfall was about $3.98 \times 10^{-3} \, M$.

Checklist

Dear learner, now it is time to check your understanding about operations on polynomial functions. Read each question and put a mark (" $\checkmark$ ") in the box that is the appropriate choice to you.

	Yes	No
• Can you apply the formula $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ to calculate compound interest?	☐	☐
• Can you apply the formula $P(t) = P_0 e^{kt}, k > 0$ to calculate population growth?	☐	☐
• Can you apply the formula $pH = -\log[H^+]$ to identify the acidity of a solution?	☐	☐

If your response to the above check list questions is positive, proceed to do the self-test exercise given below and check your answer at the end of this module. If not please review the section again and again until you think you have clear understanding about it.

Self-Test Exercise 7

1. Suppose that birr 5000 is deposited in to an account paying 5% annual interest compounded quarterly, how much money will be in the account after 4 years?

2. How much money could be deposited into an account paying 7% annual interst compunded monthly to have birr 20, 000 in the account after 5 years?

3. A Population of cockroaches grows exponentially. There are 5 initially and 60 after 2 days. How many are there after 10 days?

4. Suppose the *pH* reading of a mango juice is 4.5. Calculate the hydrogen ion concentration $[H^+]$ of the juice.

5. Suppose the hydrogin concentration reading of milk is $3.16 \times 10^{-7} \, M$. Calculate the pH of the substance and determine whether it is acidic or basic.

Key terms

- ✓ Asymptote
- ✓ antilogarithm
- ✓ base
- ✓ characteristics
- ✓ common logarithm
- ✓ exponent

- ✓ exponential equation
- ✓ exponential function
- ✓ logarithm
- ✓ logarithm of a number
- ✓ logarithmic equation
- ✓ logarithmic function

- ✓ mantissa
- ✓ natural logarithm
- ✓ logarithmic equation
- ✓ power

Unit Summary

1. For a natural number n and a real number a, the power a^n, read "the nth power of a" or "a raised to n", is defined as follows:

$$a^n = \underbrace{a \times a \times a \times \ \cdots \ \times a}_{n \ factors}$$

In the symbol a^n, a is called **the base** and n is called **the exponent**.

2. Laws of exponenets: For a real number a and natural numbers m and n,

a) $a^m \times a^n = a^{m+n}$.

b) $\dfrac{a^m}{a^n} = a^{m-n}, a \neq 0$.

c) $(a^m)^n = a^{mn}$.

d) $(a \times b)^n = a^n \times b^n$.

e) $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}, b \neq 0$.

3. $a^0 = 1, a \neq 0$.

4. Laws/properties of logarithms: For any positive numbers x and y and $a > 0$ and $a \neq 1$,

a) $\log_a xy = \log_a x + \log_a y$.

b) $\log_a \left(\frac{x}{y} \right) = \log_a x - \log_a y$.

c) $\log_a x^r = r\log_a x$.

d) $\log_a x = \frac{\log_b x}{\log_b a}$.

e) $\log_a a = 1$.

f) $\log_a 1 = 0$.

g) $a^{\log_a x} = x$.

5. The exponential function f with base a is denoted by $f(x) = a^x$, where $a > 0, a \neq 1$, and x is any real number.

6. The natural exponential function is $f(x) = e^x$, where e is the constant $2.718281828\ldots$ Its graph has the same basic characteristics as the graph of $f(x) = a^x$.

7. The graphs of the exponential functions $f(x) = a^x$ and $f(x) = a^{-x}$ have one y-intercept $(0,1)$, one horizontal asymptote (the x-axis), and are continuos.

8. For $x > 0, a > 0, and\ a \neq 1, y = \log_a x$ if and only if $x = a^y$. The function $\quad f(x) = \log_a x$ is called the logarithmic function with base a.

9. The graphs of the logarithmic function $f(x) = \log_a x$ where $a > 1$, is the graph of the inverse of $f(x) = a^x$.

For $x > 0, y = \ln x$ if and only if $x = e^y$. The function given by $f(x) = \log_e x = \ln x$ is called the natural logarithmic function. Its graph has the same basic characterstics as the graph of $f(x) = \log_a x$. They have the same x-intercept $(1,0)$, the same verticall asymptote (the y-axis), and are continuos

Review Exercises

I. Multiple Choice problems

1. Let $f(x) = \log_4 x$, then which of the following is not true?

   A. $f(4) = 1$ **B.** $f(1) = 0$ **C.** $f\left(\frac{1}{4}\right) = -1$ D. $f(16) = 3$

2. Which of the following is not true.

   A. $\log_{\left(\frac{1}{27}\right)} 3 = -\frac{1}{3}$ **B.** $\log_{\sqrt{2}} 16 = 4$ **C.** $\log_{\left(\frac{1}{5}\right)} 5 = -1$ **D.** $\log_{0.001} 0.1 = \frac{1}{3}$

3. The value of $\log_5 20 + \log_5 \left(\frac{125}{4}\right) - \log_5 \left(\frac{1}{25}\right)$ is:

   A. 6   **B.** -6   **C.** 3   **D.** -4

4. Which of the following is not true?

   A. If $\log_{\sqrt{2}} x = 6$, then $x = 8$.   **B.** If $\log_x 6 = \frac{1}{2}, x = 36$.

C. $\log_{\left(\frac{1}{2}\right)} 2 = x$, then $x = -1$.　　　　　**D.** $\log_4 0.125 = x$, then $x = \frac{3}{2}$.

5. What is the solution of the equation $9^{1-2x} = \sqrt{27}$?

　a. $\frac{1}{8}$　　　　　　　　　　　　　**C.** $-\frac{1}{8}$

　b. $\frac{1}{4}$　　　　　　　　　　　　　**D.** $\frac{1}{2}$

6. Which one of the following is an exponential function?

　A. $f(x) = \left(\frac{1}{3}\right)^x$　　　　　　　　　　**B.** $g(x) = \sqrt[5]{x}$

　C. $h(x) = 4^{\frac{1}{x}}$　　　　　　　　　　　**D.** $f(x) = x^5$

7. Which of the following is not true about $f(x) = a^x$ and $g(x) = b^x$

　where $0 < b < 1$ and $a > 0$?

　　A. Both have the same y-intercept.　　　**B.** Both have the same domain.

　　C. Both have the same range.　　　　　　**D.** Both are increasing.

II. Short answer Problems: Give the most simplified answer.

1. The exponential statement of $\log_6 36 = 2$ is _____.

2. The logarithmic statement of $5^{-3} = \frac{1}{125}$ is _____.

3. If $p = \log_6 25$ and $q = \log_6 2$, then the value of $\log_6 200$ in terms p and q is _____.

4. Simplified form of $\sqrt{2^{10x}} \cdot \left(\frac{1}{8}\right)^{x-2} \cdot 4^{-x-1}$

5. Find the value of x such that

　a) $\left(\frac{1}{16}\right)^{2x-1} = 8^{5-x}$　　　　　　　**b)** $\left(\frac{1}{2}\right)^{3x+2} \cdot 64^{3x-1} = \frac{1}{256}$

　c) $2\left(\frac{1}{4}\right)^x - \frac{1}{2^x} = 6$　　　　　　　**d)** $2^{1+x} = 6$

6. State the universe and find the value of x such that

　a) $15\log_2(3 - 2x) = 60$　　　　　　**b)** $\log_5 x + \log_5(x - 5) = \frac{1}{2}\log_5 36$

　c) $\log_2(x + 1) = 3 + \log_2(x - 4)$　　　**d)** $6(\log x)^2 = \log x + 1$

III. Workout Problems: Show all the necessary steps.

1. Solve each of the following equations.

　a) $4^{x+1} = 8 - 2^{x+2}$　　　　　　**b)** $\frac{3^x}{27^{2x-1}} = 9^{x+6}$

　c) $2^{4x+1} = \log_2 64 + 2^{2(x+1)}$　　　**d)** $\frac{16^{(5-2x)} \times 32^{(x-2)}}{4^{(5x-1)}} = \frac{1}{2}$

2. State the universe and solve each of the following equations.

　a) $\log_3(2x + 5) = 1 - \log_3 x$　　　　　　**b)** $2\log(2x - 2) = \log(4x^2 + 3)$

　c) $\log_2(8x - 1) - 2\log_2(x + 1) = 3 - \log_2(x + 4)$　**d)** $2\log_2\left(\frac{x}{2}\right) + \log_2 \sqrt{x} = 8$

e) $2\log_2 x + \log_2(x-1) - \log_2(5x+4) = 1$ f) $\dfrac{1}{\log_x \sqrt{3}} = \log_3(6-x)$

3. Given $\log_a 2 = m$ and $\log_a 5 = n$. Then express the following interms of m and n.

a) $\log_a 20$ b) $\log_a 0.4$ c) $\log_a 3.2$ d) $\log_4 5$

4. For the function given below, find the x-intercept, the y-intercept, the asymptot, the

domain, the range and sketch its graph.

a) $h(x) = 3 - 3^x$ b) $g(x) = 1 + \log_{\left(\frac{1}{2}\right)}(x+1)$

5. A culture contains 10,000 bacteria initially. After an hour, the bacteria count is 30,000.

a) Find the doubling period.   b) Find the number of bacteria after 3 .

6. Determine the final account balance of an investment if birr 20,000 is invested at an interest of

5.25% compounded semi anually for 10 years.

7. Determine the amount of interest earned if birr 3000 is invested in a bank paying 7.75% annual

interest coumpounded monthly for four years.

Module 1 Written Assignment (Model)

General instruction: - Dear distance learner this is a model for the assignment that will be given by your center which will account 30 % of your total assessment. The assignment has three parts. Multiple choice problems, short answer problems and workout problems. Do each of the according to the given instruction.

I. **Choose the correct answer for each of the following questions**

1. If f is polynomial function given by $f(x) = 2x^4 - 9x^3 + x^2 - x + 6$, then this function has a root

between _____.

    A)   1 and 2   B) -1 and 0

    C) -1 and -2 D)   0 and 1

2. If a polynomial function f is given by $f(x) = (2x+3)(5-3x)(x^2+1)$, then which one of the

following statement is correct about $f(x)$?

    A.   The degree of $p(x)$ is 3.

    B. $x = -1$ is the zero of $f(x)$.

    C.   The leading coefficient of $p(x)$ is -6.

    D.   The constant term of $p(x)$ is 10

3. Which one of the following is a true statement?

    A. $x - 1$ is the factor of $x^n + 1$, for a natural number n.

    B.   If you divide $2x^3 - 4x^2 + x + 5$ by $x + 2$, then the remainder is -29.

    C. $x - 2$ is the factor of $x^4 - 2x^3 + 3x^2 - 4x + 3$.

    D. If $x^4 + 4x^2 + 3$ is divided by $x - 1$; the constant term of the quotient is -5.

4. Which of the following statements is not true about the functions $f(x) = a^x$ and $g(x) = \log_a x$

   where $a > 0$ and $a \neq 1$?

    A. The range of f is the same as the domain of g

    B. The domain of f is the same as the range of g

    C. The range of $f(x)$ is $\{y: y > 0\}$

    D. The domain of $g(x)$ is $\{x: x \geq 0\}$

5. Which one following statement is not true? (EGSECE **2003 E.C.**)

    A. $2^x < 3^x$ for all x > 0   **B.** $\log_2 x < \log_3 x$ for all x > 1

    C. $2^{-x} > 3^{-x}$ for all x > 0   **D.** $\log_{\frac{1}{2}} x < \log_{\frac{1}{3}} x$ for all x > 1

6. The solution set of $\log_2[2 + \log_3(x + 3)] = 0$ is equal to

    A. $\left\{-\frac{8}{3}\right\}$ **B.** $\left\{\frac{1}{3}\right\}$ **C.** $\{\ \}$ **D.** $\{-1\}$

II. Give short answer for each of the following question

1. Find the values of

    a) $\log_2 6 + \log_2\left(\frac{1}{12}\right)$ **b)** $\log_3 10 + \log_3\left(\frac{6}{5}\right) + \log_3\left(\frac{9}{4}\right)$ **c)** $\frac{1}{2}\log_4 8 + \log_4 \sqrt{2}$

2. Find the values of the following common logarithms.

    a) $\log\left(\frac{1}{\sqrt[5]{10}}\right)$ **b)** $\log\left(\frac{10^m}{10^n}\right)$

3. In each of the following, find the quotient and the remainder when $p(x)$ is divided by $q(x)$

    a. $p(x) = x^5 + 2x^4 - x^3 + 5x^2 - x - 2;\ \ q(x) = x^3 + 1$

    b. $p(x) = 2x^3 - x^2 + 2x - 1:\ q(x) = 2x + 1$

III. Workout problems: Show all the necessary steps

1. Solve for x.

    a) $3\left(\frac{27}{8}\right)^{\frac{2}{3}x} = 2\left(\frac{32}{243}\right)^{2x}$ **b)** $\left(\frac{1}{2}\right)^{6x} - 2^3\left(\frac{1}{8}\right)^x = 128$

2. State the universe and solve for x.

    a) $\log_2\left(1 + \frac{1}{x}\right) = 3$ **b)** $\log_2 2 + \log_2(x + 2) - \log_2(3x - 5) = 3$

3. Sketch the graph of the relation R and find its domain and range were

   $R = \{(x, y): y \leq -x + 2 \text{ and } y > x - 2\}$.

4. For the function given below, find the x-intercept, the y-intercept, the asymptot, the

   domain, the range and sketch its graph.

    a) $f(x) = -2 + 2^x$ **b)** $f(x) = \log_2(x + 2)$

5. When the polynomial $f(x) = a(2x + 1)^2 + b(x - 2)^2$ is divided by $x + 1$ the remainder is

   -10 and $f(1) = 10$. Then find the values of a and b.

	Logarithm table										Mean Difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	0	0.0043	0.0086	0.0128	0.017	0.0212	0.0253	0.0294	0.0334	0.0374	4	8	12	17	21	25	29	33	37
1.1	0.0414	0.0453	0.0492	0.0531	0.0569	0.0607	0.0645	0.0682	0.0719	0.0755	4	8	11	15	19	23	26	30	34
1.2	0.0792	0.0828	0.0864	0.0899	0.0934	0.0969	0.1004	0.1038	0.1072	0.1106	3	7	10	14	17	21	24	28	31
1.3	0.1139	0.1173	0.1206	0.1239	0.1271	0.1303	0.1335	0.1367	0.1399	0.143	3	6	10	13	16	19	23	26	29
1.4	0.1461	0.1492	0.1523	0.1553	0.1584	0.1614	0.1644	0.1673	0.1703	0.1732	3	6	9	12	15	18	21	24	27
1.5	0.1761	0.179	0.1818	0.1847	0.1875	0.1903	0.1931	0.1959	0.1987	0.2014	3	6	8	11	14	17	20	22	25
1.6	0.2041	0.2068	0.2095	0.2122	0.2148	0.2175	0.2201	0.2227	0.2253	0.2279	3	5	8	11	13	16	18	21	24
1.7	0.2304	0.233	0.2355	0.238	0.2405	0.243	0.2455	0.248	0.2504	0.2529	2	5	7	10	12	15	17	20	22
1.8	0.2553	0.2577	0.2601	0.2625	0.2648	0.2672	0.2695	0.2718	0.2742	0.2765	2	5	7	9	12	14	16	19	21
1.9	0.2788	0.281	0.2833	0.2856	0.2878	0.29	0.2923	0.2945	0.2967	0.2989	2	4	7	9	11	13	16	18	20
2.0	0.301	0.3032	0.3054	0.3075	0.3096	0.3118	0.3139	0.316	0.3181	0.3201	2	4	6	8	11	13	15	17	19
2.1	0.3222	0.3243	0.3263	0.3284	0.3304	0.3324	0.3345	0.3365	0.3385	0.3404	2	4	6	8	10	12	14	16	18
2.2	0.3424	0.3444	0.3464	0.3483	0.3502	0.3522	0.3541	0.356	0.3579	0.3598	2	4	6	8	10	12	14	15	17
2.3	0.3617	0.3636	0.3655	0.3674	0.3692	0.3711	0.3729	0.3747	0.3766	0.3784	2	4	6	7	9	11	13	15	17
2.4	0.3802	0.382	0.3838	0.3856	0.3874	0.3892	0.3909	0.3927	0.3945	0.3962	2	4	5	7	9	11	12	14	16
2.5	0.3979	0.3997	0.4014	0.4031	0.4048	0.4065	0.4082	0.4099	0.4116	0.4133	2	3	5	7	9	10	12	14	15
2.6	0.415	0.4166	0.4183	0.42	0.4216	0.4232	0.4249	0.4265	0.4281	0.4298	2	3	5	7	8	10	11	13	15
2.7	0.4314	0.433	0.4346	0.4362	0.4378	0.4393	0.4409	0.4425	0.444	0.4456	2	3	5	6	8	9	11	13	14
2.8	0.4472	0.4487	0.4502	0.4518	0.4533	0.4548	0.4564	0.4579	0.4594	0.4609	2	3	5	6	8	9	11	12	14
2.9	0.4624	0.4639	0.4654	0.4669	0.4683	0.4698	0.4713	0.4728	0.4742	0.4757	1	3	4	6	7	9	10	12	13
3.0	0.4771	0.4786	0.48	0.4814	0.4829	0.4843	0.4857	0.4871	0.4886	0.49	1	3	4	6	7	9	10	11	13
3.1	0.4914	0.4928	0.4942	0.4955	0.4969	0.4983	0.4997	0.5011	0.5024	0.5038	1	3	4	6	7	8	10	11	12
3.2	0.5051	0.5065	0.5079	0.5092	0.5105	0.5119	0.5132	0.5145	0.5159	0.5172	1	3	4	5	7	8	9	11	12
3.3	0.5185	0.5198	0.5211	0.5224	0.5237	0.525	0.5263	0.5276	0.5289	0.5302	1	3	4	5	6	8	9	10	12
3.4	0.5315	0.5328	0.534	0.5353	0.5366	0.5378	0.5391	0.5403	0.5416	0.5428	1	3	4	5	6	8	9	10	11
3.5	0.5441	0.5453	0.5465	0.5478	0.549	0.5502	0.5514	0.5527	0.5539	0.5551	1	2	4	5	6	7	9	10	11
3.6	0.5563	0.5575	0.5587	0.5599	0.5611	0.5623	0.5635	0.5647	0.5658	0.567	1	2	4	5	6	7	8	10	11
3.7	0.5682	0.5694	0.5705	0.5717	0.5729	0.574	0.5752	0.5763	0.5775	0.5786	1	2	3	5	6	7	8	9	10
3.8	0.5798	0.5809	0.5821	0.5832	0.5843	0.5855	0.5866	0.5877	0.5888	0.5899	1	2	3	5	6	7	8	9	10
3.9	0.5911	0.5922	0.5933	0.5944	0.5955	0.5966	0.5977	0.5988	0.5999	0.601	1	2	3	4	5	7	8	9	10
4.0	0.6021	0.6031	0.6042	0.6053	0.6064	0.6075	0.6085	0.6096	0.6107	0.6117	1	2	3	4	5	6	8	9	10
4.1	0.6128	0.6138	0.6149	0.616	0.617	0.618	0.6191	0.6201	0.6212	0.6222	1	2	3	4	5	6	7	8	9
4.2	0.6232	0.6243	0.6253	0.6263	0.6274	0.6284	0.6294	0.6304	0.6314	0.6325	1	2	3	4	5	6	7	8	9
4.3	0.6335	0.6345	0.6355	0.6365	0.6375	0.6385	0.6395	0.6405	0.6415	0.6425	1	2	3	4	5	6	7	8	9
4.4	0.6435	0.6444	0.6454	0.6464	0.6474	0.6484	0.6493	0.6503	0.6513	0.6522	1	2	3	4	5	6	7	8	9
4.5	0.6532	0.6542	0.6551	0.6561	0.6571	0.658	0.659	0.6599	0.6609	0.6618	1	2	3	4	5	6	7	8	9
4.6	0.6628	0.6637	0.6646	0.6656	0.6665	0.6675	0.6684	0.6693	0.6702	0.6712	1	2	3	4	5	6	7	7	8
4.7	0.6721	0.673	0.6739	0.6749	0.6758	0.6767	0.6776	0.6785	0.6794	0.6803	1	2	3	4	5	5	6	7	8
4.8	0.6812	0.6821	0.683	0.6839	0.6848	0.6857	0.6866	0.6875	0.6884	0.6893	1	2	3	4	4	5	6	7	8
4.9	0.6902	0.6911	0.692	0.6928	0.6937	0.6946	0.6955	0.6964	0.6972	0.6981	1	2	3	4	4	5	6	7	8
5.0	0.699	0.6998	0.7007	0.7016	0.7024	0.7033	0.7042	0.705	0.7059	0.7067	1	2	3	3	4	5	6	7	8
5.1	0.7076	0.7084	0.7093	0.7101	0.711	0.7118	0.7126	0.7135	0.7143	0.7152	1	2	3	3	4	5	6	7	8
5.2	0.716	0.7168	0.7177	0.7185	0.7193	0.7202	0.721	0.7218	0.7226	0.7235	1	2	2	3	4	5	6	7	7
5.3	0.7243	0.7251	0.7259	0.7267	0.7275	0.7284	0.7292	0.73	0.7308	0.7316	1	2	2	3	4	5	6	6	7
5.4	0.7324	0.7332	0.734	0.7348	0.7356	0.7364	0.7372	0.738	0.7388	0.7396	1	2	2	3	4	5	6	6	7
5.5	0.7404	0.7412	0.7419	0.7427	0.7435	0.7443	0.7451	0.7459	0.7466	0.7474	1	2	2	3	4	5	5	6	7

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
				Logarithm table											Mean Difference				
5.6	0.7482	0.749	0.7497	0.7505	0.7513	0.752	0.7528	0.7536	0.7543	0.7551	1	2	2	3	4	5	5	6	7
5.7	0.7559	0.7566	0.7574	0.7582	0.7589	0.7597	0.7604	0.7612	0.7619	0.7627	1	2	2	3	4	5	5	6	7
5.8	0.7634	0.7642	0.7649	0.7657	0.7664	0.7672	0.7679	0.7686	0.7694	0.7701	1	1	2	3	4	4	5	6	7
5.9	0.7709	0.7716	0.7723	0.7731	0.7738	0.7745	0.7752	0.776	0.7767	0.7774	1	1	2	3	4	4	5	6	7
6.0	0.7782	0.7789	0.7796	0.7803	0.781	0.7818	0.7825	0.7832	0.7839	0.7846	1	1	2	3	4	4	5	6	6
6.1	0.7853	0.786	0.7868	0.7875	0.7882	0.7889	0.7896	0.7903	0.791	0.7917	1	1	2	3	4	4	5	6	6
6.2	0.7924	0.7931	0.7938	0.7945	0.7952	0.7959	0.7966	0.7973	0.798	0.7987	1	1	2	3	3	4	5	6	6
6.3	0.7993	0.8	0.8007	0.8014	0.8021	0.8028	0.8035	0.8041	0.8048	0.8055	1	1	2	3	3	4	5	5	6
6.4	0.8062	0.8069	0.8075	0.8082	0.8089	0.8096	0.8102	0.8109	0.8116	0.8122	1	1	2	3	3	4	5	5	6
6.5	0.8129	0.8136	0.8142	0.8149	0.8156	0.8162	0.8169	0.8176	0.8182	0.8189	1	1	2	3	3	4	5	5	6
6.6	0.8195	0.8202	0.8209	0.8215	0.8222	0.8228	0.8235	0.8241	0.8248	0.8254	1	1	2	3	3	4	5	5	6
6.7	0.8261	0.8267	0.8274	0.828	0.8287	0.8293	0.8299	0.8306	0.8312	0.8319	1	1	2	3	3	4	5	5	6
6.8	0.8325	0.8331	0.8338	0.8344	0.8351	0.8357	0.8363	0.837	0.8376	0.8382	1	1	2	3	3	4	4	5	6
6.9	0.8388	0.8395	0.8401	0.8407	0.8414	0.842	0.8426	0.8432	0.8439	0.8445	1	1	2	2	3	4	4	5	6
7.0	0.8451	0.8457	0.8463	0.847	0.8476	0.8482	0.8488	0.8494	0.85	0.8506	1	1	2	2	3	4	4	5	6
7.1	0.8513	0.8519	0.8525	0.8531	0.8537	0.8543	0.8549	0.8555	0.8561	0.8567	1	1	2	2	3	4	4	5	5
7.2	0.8573	0.8579	0.8585	0.8591	0.8597	0.8603	0.8609	0.8615	0.8621	0.8627	1	1	2	2	3	4	4	5	5
7.3	0.8633	0.8639	0.8645	0.8651	0.8657	0.8663	0.8669	0.8675	0.8681	0.8686	1	1	2	2	3	4	4	5	5
7.4	0.8692	0.8698	0.8704	0.871	0.8716	0.8722	0.8727	0.8733	0.8739	0.8745	1	1	2	2	3	4	4	5	5
7.5	0.8751	0.8756	0.8762	0.8768	0.8774	0.8779	0.8785	0.8791	0.8797	0.8802	1	1	2	2	3	3	4	5	5
7.6	0.8808	0.8814	0.882	0.8825	0.8831	0.8837	0.8842	0.8848	0.8854	0.8859	1	1	2	2	3	3	4	5	5
7.7	0.8865	0.8871	0.8876	0.8882	0.8887	0.8893	0.8899	0.8904	0.891	0.8915	1	1	2	2	3	3	4	4	5
7.8	0.8921	0.8927	0.8932	0.8938	0.8943	0.8949	0.8954	0.896	0.8965	0.8971	1	1	2	2	3	3	4	4	5
7.9	0.8976	0.8982	0.8987	0.8993	0.8998	0.9004	0.9009	0.9015	0.902	0.9025	1	1	2	2	3	3	4	4	5
8.0	0.9031	0.9036	0.9042	0.9047	0.9053	0.9058	0.9063	0.9069	0.9074	0.9079	1	1	2	2	3	3	4	4	5
8.1	0.9085	0.909	0.9096	0.9101	0.9106	0.9112	0.9117	0.9122	0.9128	0.9133	1	1	2	2	3	3	4	4	5
8.2	0.9138	0.9143	0.9149	0.9154	0.9159	0.9165	0.917	0.9175	0.918	0.9186	1	1	2	2	3	3	4	4	5
8.3	0.9191	0.9196	0.9201	0.9206	0.9212	0.9217	0.9222	0.9227	0.9232	0.9238	1	1	2	2	3	3	4	4	5
8.4	0.9243	0.9248	0.9253	0.9258	0.9263	0.9269	0.9274	0.9279	0.9284	0.9289	1	1	2	2	3	3	4	4	5
8.5	0.9294	0.9299	0.9304	0.9309	0.9315	0.932	0.9325	0.933	0.9335	0.934	1	1	2	2	3	3	4	4	5
8.6	0.9345	0.935	0.9355	0.936	0.9365	0.937	0.9375	0.938	0.9385	0.939	1	1	2	2	3	3	4	4	5
8.7	0.9395	0.94	0.9405	0.941	0.9415	0.942	0.9425	0.943	0.9435	0.944	0	1	1	2	2	3	3	4	4
8.8	0.9445	0.945	0.9455	0.946	0.9465	0.9469	0.9474	0.9479	0.9484	0.9489	0	1	1	2	2	3	3	4	4
8.9	0.9494	0.9499	0.9504	0.9509	0.9513	0.9518	0.9523	0.9528	0.9533	0.9538	0	1	1	2	2	3	3	4	4
9.0	0.9542	0.9547	0.9552	0.9557	0.9562	0.9566	0.9571	0.9576	0.9581	0.9586	0	1	1	2	2	3	3	4	4
9.1	0.959	0.9595	0.96	0.9605	0.9609	0.9614	0.9619	0.9624	0.9628	0.9633	0	1	1	2	2	3	3	4	4
9.2	0.9638	0.9643	0.9647	0.9652	0.9657	0.9661	0.9666	0.9671	0.9675	0.968	0	1	1	2	2	3	3	4	4
9.3	0.9685	0.9689	0.9694	0.9699	0.9703	0.9708	0.9713	0.9717	0.9722	0.9727	0	1	1	2	2	3	3	4	4
9.4	0.9731	0.9736	0.9741	0.9745	0.975	0.9754	0.9759	0.9763	0.9768	0.9773	0	1	1	2	2	3	3	4	4
9.5	0.9777	0.9782	0.9786	0.9791	0.9795	0.98	0.9805	0.9809	0.9814	0.9818	0	1	1	2	2	3	3	4	4
9.6	0.9823	0.9827	0.9832	0.9836	0.9841	0.9845	0.985	0.9854	0.9859	0.9863	0	1	1	2	2	3	3	4	4
9.7	0.9868	0.9872	0.9877	0.9881	0.9886	0.989	0.9894	0.9899	0.9903	0.9908	0	1	1	2	2	3	3	4	4
9.8	0.9912	0.9917	0.9921	0.9926	0.993	0.9934	0.9939	0.9943	0.9948	0.9952	0	1	1	2	2	3	3	4	4
9.9	0.9956	0.9961	0.9965	0.9969	0.9974	0.9978	0.9983	0.9987	0.9991	0.9996	0	1	1	2	2	3	3	3	4

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
	Anti Logarithm table										Mean Difference								
0.00	1	1.002	1.005	1.007	1.009	1.012	1.014	1.016	1.019	1.021	0	0	1	1	1	1	2	2	2
0.01	1.023	1.026	1.028	1.03	1.033	1.035	1.038	1.04	1.042	1.045	0	0	1	1	1	1	2	2	2
0.02	1.047	1.05	1.052	1.054	1.057	1.059	1.062	1.064	1.067	1.069	0	0	1	1	1	1	2	2	2
0.03	1.072	1.074	1.076	1.079	1.081	1.084	1.086	1.089	1.091	1.094	0	0	1	1	1	1	2	2	2
0.04	1.096	1.099	1.102	1.104	1.107	1.109	1.112	1.114	1.117	1.119	0	1	1	1	1	2	2	2	2
0.05	1.122	1.125	1.127	1.13	1.132	1.135	1.138	1.14	1.143	1.146	0	1	1	1	1	2	2	2	2
0.06	1.148	1.151	1.153	1.156	1.159	1.161	1.164	1.167	1.169	1.172	0	1	1	1	1	2	2	2	2
0.07	1.175	1.178	1.18	1.183	1.186	1.189	1.191	1.194	1.197	1.199	0	1	1	1	1	2	2	2	2
0.08	1.202	1.205	1.208	1.211	1.213	1.216	1.219	1.222	1.225	1.227	0	1	1	1	1	2	2	2	3
0.09	1.23	1.233	1.236	1.239	1.242	1.245	1.247	1.25	1.253	1.256	0	1	1	1	1	2	2	2	3
0.10	1.259	1.262	1.265	1.268	1.271	1.274	1.276	1.279	1.282	1.285	0	1	1	1	1	2	2	2	3
0.11	1.288	1.291	1.294	1.297	1.3	1.303	1.306	1.309	1.312	1.315	0	1	1	1	2	2	2	2	3
0.12	1.318	1.321	1.324	1.327	1.33	1.334	1.337	1.34	1.343	1.346	0	1	1	1	2	2	2	2	3
0.13	1.349	1.352	1.355	1.358	1.361	1.365	1.368	1.371	1.374	1.377	0	1	1	1	2	2	2	3	3
0.14	1.38	1.384	1.387	1.39	1.393	1.396	1.4	1.403	1.406	1.409	0	1	1	1	2	2	2	3	3
0.15	1.413	1.416	1.419	1.422	1.426	1.429	1.432	1.435	1.439	1.442	0	1	1	1	2	2	2	3	3
0.16	1.445	1.449	1.452	1.455	1.459	1.462	1.466	1.469	1.472	1.476	0	1	1	1	2	2	2	3	3
0.17	1.479	1.483	1.486	1.489	1.493	1.496	1.5	1.503	1.507	1.51	0	1	1	1	2	2	2	3	3
0.18	1.514	1.517	1.521	1.524	1.528	1.531	1.535	1.538	1.542	1.545	0	1	1	1	2	2	2	3	3
0.19	1.549	1.552	1.556	1.56	1.563	1.567	1.57	1.574	1.578	1.581	0	1	1	1	2	2	3	3	3
0.20	1.585	1.589	1.592	1.596	1.6	1.603	1.607	1.611	1.614	1.618	0	1	1	1	2	2	3	3	3
0.21	1.622	1.626	1.629	1.633	1.637	1.641	1.644	1.648	1.652	1.656	0	1	1	2	2	2	3	3	3
0.22	1.66	1.663	1.667	1.671	1.675	1.679	1.683	1.687	1.69	1.694	0	1	1	2	2	2	3	3	3
0.23	1.698	1.702	1.706	1.71	1.714	1.718	1.722	1.726	1.73	1.734	0	1	1	2	2	2	3	3	4
0.24	1.738	1.742	1.746	1.75	1.754	1.758	1.762	1.766	1.77	1.774	0	1	1	2	2	2	3	3	4
0.25	1.778	1.782	1.786	1.791	1.795	1.799	1.803	1.807	1.811	1.816	0	1	1	2	2	2	3	3	4
0.26	1.82	1.824	1.828	1.832	1.837	1.841	1.845	1.849	1.854	1.858	0	1	1	2	2	3	3	3	4
0.27	1.862	1.866	1.871	1.875	1.879	1.884	1.888	1.892	1.897	1.901	0	1	1	2	2	3	3	3	4
0.28	1.905	1.91	1.914	1.919	1.923	1.928	1.932	1.936	1.941	1.945	0	1	1	2	2	3	3	4	4
0.29	1.95	1.954	1.959	1.963	1.968	1.972	1.977	1.982	1.986	1.991	0	1	1	2	2	3	3	4	4
0.30	1.995	2	2.004	2.009	2.014	2.018	2.023	2.028	2.032	2.037	0	1	1	2	2	3	3	4	4
0.31	2.042	2.046	2.051	2.056	2.061	2.065	2.07	2.075	2.08	2.084	0	1	1	2	2	3	3	4	4
0.32	2.089	2.094	2.099	2.104	2.109	2.113	2.118	2.123	2.128	2.133	0	1	1	2	2	3	3	4	4
0.33	2.138	2.143	2.148	2.153	2.158	2.163	2.168	2.173	2.178	2.183	0	1	1	2	2	3	3	4	4
0.34	2.188	2.193	2.198	2.203	2.208	2.213	2.218	2.223	2.228	2.234	1	1	2	2	3	3	4	4	5
0.35	2.239	2.244	2.249	2.254	2.259	2.265	2.27	2.275	2.28	2.286	1	1	2	2	3	3	4	4	5
0.36	2.291	2.296	2.301	2.307	2.312	2.317	2.323	2.328	2.333	2.339	1	2	2	3	3	4	4	5
0.37	2.344	2.35	2.355	2.36	2.366	2.371	2.377	2.382	2.388	2.393	1	1	2	2	3	3	4	4	5
0.38	2.399	2.404	2.41	2.415	2.421	2.427	2.432	2.438	2.443	2.449	1	1	2	2	3	3	4	4	5
0.39	2.455	2.46	2.466	2.472	2.477	2.483	2.489	2.495	2.5	2.506	1	1	2	2	3	3	4	5	5
0.40	2.512	2.518	2.523	2.529	2.535	2.541	2.547	2.553	2.559	2.564	1	1	2	2	3	4	4	5	5
0.41	2.57	2.576	2.582	2.588	2.594	2.6	2.606	2.612	2.618	2.624	1	1	2	2	3	4	4	5	5
0.42	2.63	2.636	2.642	2.649	2.655	2.661	2.667	2.673	2.679	2.685	1	1	2	2	3	4	4	5	6
0.43	2.692	2.698	2.704	2.71	2.716	2.723	2.729	2.735	2.742	2.748	1	1	2	3	3	4	4	5	6
0.44	2.754	2.761	2.767	2.773	2.78	2.786	2.793	2.799	2.805	2.812	1	1	2	3	3	4	4	5	6
0.45	2.818	2.825	2.831	2.838	2.844	2.851	2.858	2.864	2.871	2.877	1	1	2	3	3	4	5	5	6
0.46	2.884	2.891	2.897	2.904	2.911	2.917	2.924	2.931	2.938	2.944	1	1	2	3	3	4	5	5	6
0.47	2.951	2.958	2.965	2.972	2.979	2.985	2.992	2.999	3.006	3.013	1	1	2	3	3	4	5	5	6
0.48	3.02	3.027	3.034	3.041	3.048	3.055	3.062	3.069	3.076	3.083	1	1	2	3	4	4	5	6	6
0.49	3.09	3.097	3.105	3.112	3.119	3.126	3.133	3.141	3.148	3.155	1	1	2	3	4	4	5	6	6
0.50	3.162	3.17	3.177	3.184	3.192	3.199	3.206	3.214	3.221	3.228	1	1	2	3	4	4	5	6	7

	Anti Logarithm table										Mean Difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.51	3.236	3.243	3.251	3.258	3.266	3.273	3.281	3.289	3.296	3.304	1	2	2	3	4	5	5	6	7
0.52	3.311	3.319	3.327	3.334	3.342	3.35	3.357	3.365	3.373	3.381	1	2	2	3	4	5	5	6	7
0.53	3.388	3.396	3.404	3.412	3.42	3.428	3.436	3.443	3.451	3.459	1	2	2	3	4	5	6	6	7
0.54	3.467	3.475	3.483	3.491	3.499	3.508	3.516	3.524	3.532	3.54	1	2	2	3	4	5	6	6	7
0.55	3.548	3.556	3.565	3.573	3.581	3.589	3.597	3.606	3.614	3.622	1	2	2	3	4	5	6	7	7
0.56	3.631	3.639	3.648	3.656	3.664	3.673	3.681	3.69	3.698	3.707	1	2	3	3	4	5	6	7	8
0.57	3.715	3.724	3.733	3.741	3.75	3.758	3.767	3.776	3.784	3.793	1	2	3	3	4	5	6	7	8
0.58	3.802	3.811	3.819	3.828	3.837	3.846	3.855	3.864	3.873	3.882	1	2	3	4	4	5	6	7	8
0.59	3.89	3.899	3.908	3.917	3.926	3.936	3.945	3.954	3.963	3.972	1	2	3	4	5	5	6	7	8
0.60	3.981	3.99	3.999	4.009	4.018	4.027	4.036	4.046	4.055	4.064	1	2	3	4	5	6	6	7	8
0.61	4.074	4.083	4.093	4.102	4.111	4.121	4.13	4.14	4.15	4.159	1	2	3	4	5	6	7	8	9
0.62	4.169	4.178	4.188	4.198	4.207	4.217	4.227	4.236	4.246	4.256	1	2	3	4	5	6	7	8	9
0.63	4.266	4.276	4.285	4.295	4.305	4.315	4.325	4.335	4.345	4.355	1	2	3	4	5	6	7	8	9
0.64	4.365	4.375	4.385	4.395	4.406	4.416	4.426	4.436	4.446	4.457	1	2	3	4	5	6	7	8	9
0.65	4.467	4.477	4.487	4.498	4.508	4.519	4.529	4.539	4.55	4.56	1	2	3	4	5	6	7	8	9
0.66	4.571	4.581	4.592	4.603	4.613	4.624	4.634	4.645	4.656	4.667	1	2	3	4	5	6	7	9	10
0.67	4.677	4.688	4.699	4.71	4.721	4.732	4.742	4.753	4.764	4.775	1	2	3	4	5	7	8	9	10
0.68	4.786	4.797	4.808	4.819	4.831	4.842	4.853	4.864	4.875	4.887	1	2	3	4	6	7	8	9	10
0.69	4.898	4.909	4.92	4.932	4.943	4.955	4.966	4.977	4.989	5	1	2	3	5	6	7	8	9	10
0.70	5.012	5.023	5.035	5.047	5.058	5.07	5.082	5.093	5.105	5.117	1	2	4	5	6	7	8	9	11
0.71	5.129	5.14	5.152	5.164	5.176	5.188	5.2	5.212	5.224	5.236	1	2	4	5	6	7	8	10	11
0.72	5.248	5.26	5.272	5.284	5.297	5.309	5.321	5.333	5.346	5.358	1	2	4	5	6	7	9	10	11
0.73	5.37	5.383	5.395	5.408	5.42	5.433	5.445	5.458	5.47	5.483	1	3	4	5	6	8	9	10	11
0.74	5.495	5.508	5.521	5.534	5.546	5.559	5.572	5.585	5.598	5.61	1	3	4	5	6	8	9	10	12
0.75	5.623	5.636	5.649	5.662	5.675	5.689	5.702	5.715	5.728	5.741	1	3	4	5	7	8	9	10	12
0.76	5.754	5.768	5.781	5.794	5.808	5.821	5.834	5.848	5.861	5.875	1	3	4	5	7	8	9	11	12
0.77	5.888	5.902	5.916	5.929	5.943	5.957	5.97	5.984	5.998	6.012	1	3	4	5	7	8	10	11	12
0.78	6.026	6.039	6.053	6.067	6.081	6.095	6.109	6.124	6.138	6.152	1	3	4	6	7	8	10	11	13
0.79	6.166	6.18	6.194	6.209	6.223	6.237	6.252	6.266	6.281	6.295	1	3	4	6	7	9	10	11	13
0.80	6.31	6.324	6.339	6.353	6.368	6.383	6.397	6.412	6.427	6.442	1	3	4	6	7	9	10	12	13
0.81	6.457	6.471	6.486	6.501	6.516	6.531	6.546	6.561	6.577	6.592	2	3	5	6	8	9	11	12	14
0.82	6.607	6.622	6.637	6.653	6.668	6.683	6.699	6.714	6.73	6.745	2	3	5	6	8	9	11	12	14
0.83	6.761	6.776	6.792	6.808	6.823	6.839	6.855	6.871	6.887	6.902	2	3	5	6	8	9	11	13	14
0.84	6.918	6.934	6.95	6.966	6.982	6.998	7.015	7.031	7.047	7.063	2	3	5	6	8	10	11	13	15
0.85	7.079	7.096	7.112	7.129	7.145	7.161	7.178	7.194	7.211	7.228	2	3	5	7	8	10	12	13	15
0.86	7.244	7.261	7.278	7.295	7.311	7.328	7.345	7.362	7.379	7.396	2	3	5	7	8	10	12	13	15
0.87	7.413	7.43	7.447	7.464	7.482	7.499	7.516	7.534	7.551	7.568	2	3	5	7	9	10	12	14	16
0.88	7.586	7.603	7.621	7.638	7.656	7.674	7.691	7.709	7.727	7.745	2	4	5	7	9	11	12	14	16
0.89	7.762	7.78	7.798	7.816	7.834	7.852	7.87	7.889	7.907	7.925	2	4	5	7	9	11	13	14	16
0.90	7.943	7.962	7.98	7.998	8.017	8.035	8.054	8.072	8.091	8.11	2	4	6	7	9	11	13	15	17
0.91	8.128	8.147	8.166	8.185	8.204	8.222	8.241	8.26	8.279	8.299	2	4	6	8	9	11	13	15	17
0.92	8.318	8.337	8.356	8.375	8.395	8.414	8.433	8.453	8.472	8.492	2	4	6	8	10	12	14	15	17
0.93	8.511	8.531	8.551	8.57	8.59	8.61	8.63	8.65	8.67	8.69	2	4	6	8	10	12	14	16	18
0.94	8.71	8.73	8.75	8.77	8.79	8.81	8.831	8.851	8.872	8.892	2	4	6	8	10	12	14	16	18
0.95	8.913	8.933	8.954	8.974	8.995	9.016	9.036	9.057	9.078	9.099	2	4	6	8	10	12	15	17	19
0.96	9.12	9.141	9.162	9.183	9.204	9.226	9.247	9.268	9.29	9.311	2	4	6	8	11	13	15	17	19
0.97	9.333	9.354	9.376	9.397	9.419	9.441	9.462	9.484	9.506	9.528	2	4	7	9	11	13	15	17	20
0.98	9.55	9.572	9.594	9.616	9.638	9.661	9.683	9.705	9.727	9.75	2	4	7	9	11	13	16	18	20
0.99	9.772	9.795	9.817	9.84	9.863	9.886	9.908	9.931	9.954	9.977	2	5	7	9	11	14	16	18	20

Feedback to Activities

Unit 1

Activity 1.1

    a. $(5, 4), (7, 4), (7, 6), (8, 4), (8, 6) \ and \ (8, 7).$

    b. $(2, 7), (2, 9), (3, 4), (3, 6), (3, 8), (5, 4), (5, 6), (5, 8), (7, 4), (7, 6), (7, 8).$

    c. $(2, 4), (2, 6), (2, 10), (3, 6), (3, 9), (5, 10).$

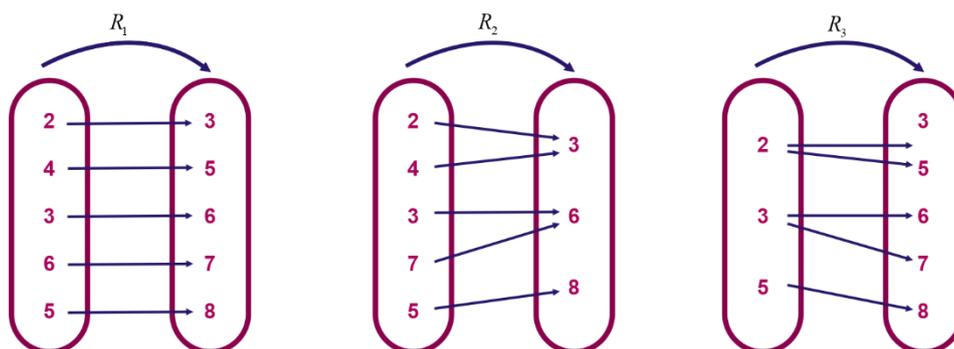   **d.** $(2, 4), (3,6) \ and \ (5, 10).$

Activity 1.2

The first elements of R is the set containing the integers 4,6 and -3 or {-3, 4, 6}.

The second elements of R is the set containing the integers 2, 3 and 4 or {2, 3, 4}.

Activity 1.3

  a. The arrow diagram:

  b. In R_1 every first element of the coordinate mapped to distinct second element of the coordinate. Whereas in R_2 two distinct first elements of the coordinate mapped to the same second coordinate and in R_3 the same first element of the coordinate mapped to two distinct second elements of the coordinate.

  c. Yes, 2 is mapped to 3 and 5, 3 are mapped to 6, 7, respectively.

Activity 1.4

f contains $(0, 0), (1, 3), (2, 6), (4,12)$ and the rule is an ordered pair (x, y) such that $y = 3x$.

Activity 1.5

    a. $(f + g)(x) = f(x) + g(x) = 2x + 4 + (-2) = 2x + 2,$

        $(f - g)(x) = 2x + 4 - (-2) = 2x + 6,$

        $(f . g) = (2x + 4)(-2) = -4x - 8$

        $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{2x + 4}{-2} = -x - 2$

b. The Domain of f = Domain of g = set of real numbers. Range of f is the set of real numbers, Range of g is $\{-2\}$.

Activity 1.6

a. $(f \cdot g)(x) = (x + 2)x^2 = x^3 + 2x^2$ and $(f \cdot g)(-1) = (-1)^3 + 2(-1)^2 = 1$.

b. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x+2}{x^2}$ and $\left(\frac{f}{g}\right)(3) = \frac{5}{9}$.

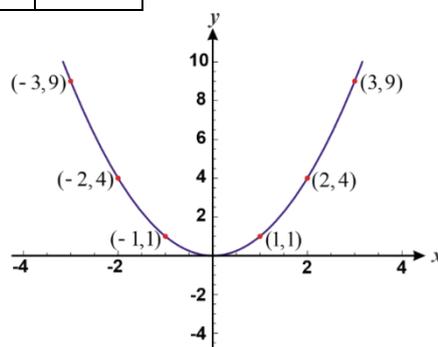**c.** $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{x^2}{x+2}$ and $\left(\frac{g}{f}\right)(3) = \frac{9}{5}$.

Activity 1.7

a.

x	-3	-2	-1	0	1	2	3
$f(x) = x^2$	9	4	1	0	1	4	9

b. The point (0,0) is x-intercept and y-intercept of the graph

c. $y = x^2$

d. The domain is the set of real numbers. i.e., $D = \{x : x \in \mathbb{R}\}$ and the range is $R = \{y | y \geq 0, y \in \mathbb{R}\}$.

The graph of $y = x^2$

Activity 1.8

For a quadratic function $f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$

a. If $a > 0$, $f(x)$ has a minimum value at the vertex $\left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right)$.

b. If $a < 0$, $f(x)$ has a maximum value at the vertex $\left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right)$.
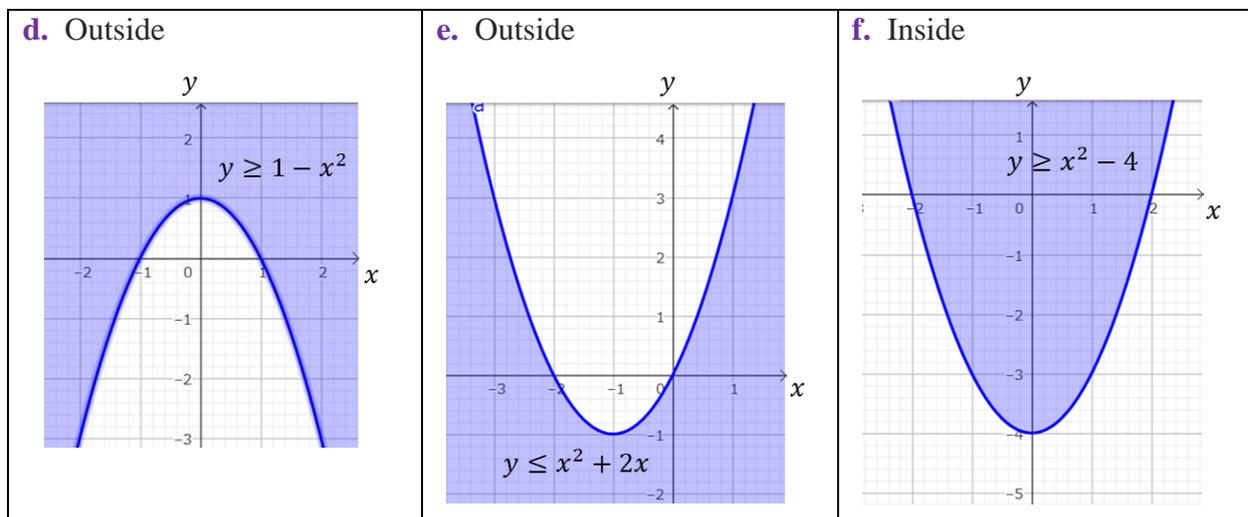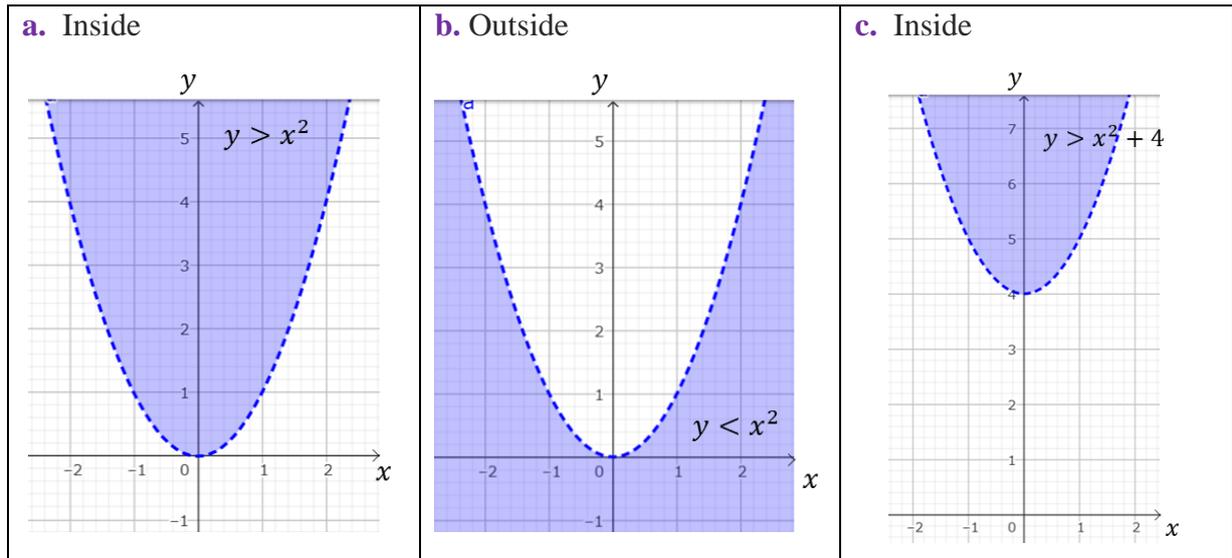
Activity 1.9

x	...	-3	-2	-1	0	1	2	3	...
y	...	9	4	1	0	1	4	4	...
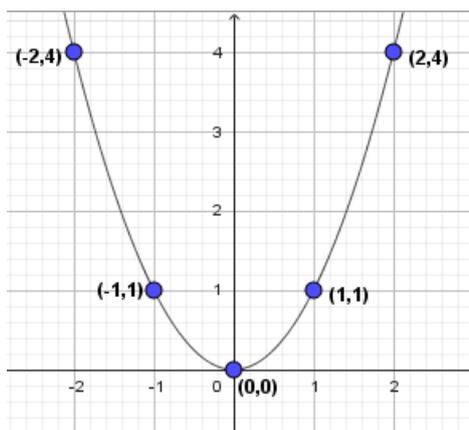
Activity 1.10

1. **a.** $(0,0)$ is the x-intercept and the y-intercept.

   b. $(0,0)$ is the x-intercept and the y-intercept.

   c. $(0,4)$ is the y-intercept and no x-intercept.

   d. $(0,1)$ is the y-intercept and $(-1,0)$ and $(1,0)$ are x-intercepts.

   e. $(-2,0)$ and $(0,0)$ are x-intercepts and $(0,0)$ is the y-intercept.

   f. $(-2,0)$ and $(2,0)$ are x-intercepts and $(0,-4)$ is the y-intercept.

2.

a. Inside	**b.** Outside	**c.** Inside
$y > x^2$	$y < x^2$	$y > x^2 + 4$

d. Outside	**e.** Outside	**f.** Inside
$y \geq 1 - x^2$	$y \leq x^2 + 2x$	$y \geq x^2 - 4$

Unit 2

Activity 2.1

    a. Linear   **b.** None

    c. Quadratic   **d.** Constant

    e. Linear   **f.** None

    g. Quadratic   **h.** None

    i. None

Activity 2.2

1. **a.** Like terms   **b.** Like terms

    c. Unlike terms   **d.** Unlike terms

    e. Unlike terms   **f.** Like terms

2.

    a. True, commutative property of addition.

    b. False, subtraction of numbers is not commutative.

    c. True, associative property of addition.

    d. True, associative property of multiplication.

    e. True, distributive property of multiplication over addition.

    f. False.

    g. True, both b and -c are multiplied by -1.

Activity 2.3

i. $\frac{97}{8} = 12 + \frac{1}{8}$ **ii.** $\frac{168}{5} = 33 + \frac{3}{5}$ **iii.** $\frac{287}{15} = 19 + \frac{2}{15}$ **iv.** $\frac{355}{11} = 32 + \frac{3}{11}$

Activity 2.4

i. $q = 17$ and $r = 3$ **ii.** $q = 50$ and $b = 5$

iii. $q = 20$ and $r = 14$ **iv.** $q = 4$ and $r = 0$

Activity 2.5

a.

$$
\begin{array}{r}
x - 3 \\
x + 2 \overline{\smash{\big)}\ x^2 - x + 3} \\
\underline{x^2 + 2x} \\
-3x + 3 \\
\underline{-3x - 6} \\
9
\end{array}
$$

$$f(c) = f(-2) = (-2)^2 - (-2) + 3 = 9$$

b.

$$
\begin{array}{r}
x^2 + 3x + 2 \\
x - 1 \overline{\smash{\big)}\ x^3 + 2x^2 - x - 5} \\
\underline{x^3 - x^2} \\
3x^2 - x - 5 \\
\underline{3x^2 - 3x} \\
2x - 5 \\
\underline{2x - 2} \\
-3
\end{array}
$$

$$f(c) = f(1) = (1)^3 + 2(1)^2 - (1) - 5 = -3$$

Activity 2.6

a. $f(1) = 0$.

b. $q(x) = x^2 + 5x + 6$ and $r(x) = 0$.

c. $f(x) = (x - 1)(x^2 + 5x + 6) + 0$.

d. Yes.

Activity 2.7

a. $x = 9$ **b.** $x = \frac{9}{8}$ **c.** $x = \frac{3}{2}$ or $x = -2$ **d.** $x = 2$ or $x = 3$

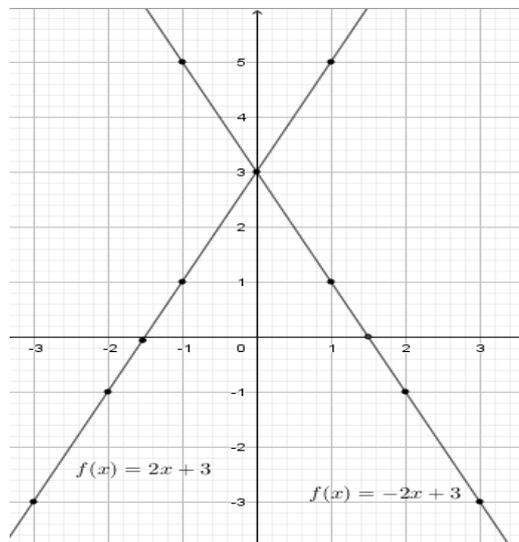**e.** $x = -1$ **f.** has no solution in the set of real numbers

Activity 2.8

a. The zeros of $f(x)$ are $0, -2$ and $\frac{3}{2}$ and they are rational numbers.

b. The zeros of $f(x)$ are $-\sqrt{2}$ and $\sqrt{3}$ and they are irrational numbers.

c. $f(x)$ has no zero in the set of real numbers.

Activity 2.9

x	-3	-2	-3/2	-1	0	1
$f(x) = 2x + 3$	-3	-1	0	1	3	5

x	-1	0	1	3/2	2	3
$g(x) = -2x + 3$	5	3	1	0	-1	-3

Unit 3

Activity 3.1

1.

    a. The base is 3; the exponent is 4.

    b. The base is -3; the exponent is 4.

   c. The base is $\frac{3}{5}$; the exponent is 5.

   d. The base is -1; the exponent is 9.

2.

   a. $(-1)^1 = -1$

   b. $(-1)^4 = (-1) \times (-1) \times (-1) \times (-1) = 1$

   c. $\left(\frac{3}{5}\right)^1 = \frac{3}{5}$

   d. $(-2)^7 = (-2) \times (-2) \times (-2) \times (-2) \times (-2) \times (-2) \times (-2) = -128$

   e. $-2^4 = -(2 \times 2 \times 2 \times 2) = -16$

   f. $(-2)^4 = (-2) \times (-2) \times (-2) \times (-2) = 16$

   g. $\left(-\frac{2}{3}\right)^4 = \left(-\frac{2}{3}\right) \times \left(-\frac{2}{3}\right) \times \left(-\frac{2}{3}\right) \times \left(-\frac{2}{3}\right) = \frac{16}{81}$

Activity 3.2

   a. $1 = \frac{9}{9} = \frac{3^2}{3^2} = 3^{2-2} = 3^0$. Hence, $3^0 = 1$.

   b. $1 = \frac{9}{9} = \frac{(-3)^2}{(-3)^2} = (-3)^{2-2} = (-3)^0$. Hence, $(-3)^0 = 1$.

   c. $1 = \frac{0.01}{0.01} = \frac{(0.1)^2}{(0.1)^2} = (0.1)^{2-2} = (0.1)^0$. Hence $(0.1)^0 = 1$.

Activity 3.3

   a. $2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}} = 2^1 = 2$

   b. $\sqrt{2} \times \sqrt{2} = 2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}} = 2^1 = 2$

   c. $2^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}} = 2^{\frac{3}{2}}$

   d. $\sqrt{2} \times \sqrt{2} \times \sqrt{2} = 2^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}} = 2^{\frac{3}{2}}$

Activity 3.4

   a. $\log 10 = \log_{10} 10 = 1$

   b. $\log_{10} 1000 = 3 \log_{10} 10 = 3$

   c. $\log_{10} 1 = = \log_{10} 10^0 = 0 \log_{10} 10 = 0$

   d. $\log_{10} 0.1 = -1$

Activity 3.5

  a)

- the number of bacteria after one hour is 2
- the number of bacteria after two hours is 4
- the number of bacteria after three hours is 8
- the number of bacteria after four hours is 16

- the number of bacteria after t hours is 2^t

b)

Time in hour(t)	0	1	2	3	4	...	t
Number of bacteria	1	2	4	8	16	...	2^t

c) The formula to calculate the number of bacteria after t hours is 2^t.

Activity 3.6

1. For positive real numbers x.

2. For negative real numbers x.

3. Yes

4. Go upward without limit.

5. Approaches the negative x-axis from above.

6. Yes. The x-axis.

Answer key to Self-Test Exercise and Unit Self-Assessment

Unit 1

Self-Test Exercises 1

1. **a.** $(-4, 3), (-2, 0), (0.2, 0.21), (0, 5)$

2. Some of the ordered pairs are $(-2, -9), (-1, -7), (0, -5), (1, -3), (2, -1), (3, 1)$.

3. **a.** $R = \left\{(0, 0), (1, -1), (2, -8), (-2, 8), \left(\frac{1}{5}, -\frac{1}{125}\right), (3, -27), (-3, 27)\right\}$.

   b. $R = \left\{(-2, 8), (1, -1), (2, -8), (3, -27), \left(-\frac{1}{3}, \frac{1}{27}\right)\right\}$

4. Domain of the relation $= \{-2, 1, 3, 4, 5\}$ and Range of the relation $= \{-3, 4, 5, 6\}$.

5.

   a. $R = \{..., (-2,6), (-1,3), (0,0), (1,-3), (2,-6), (3, -9), ...\}$.

   Domain of the relation $R = \{..., -2, -1, 0, 1, 2, 3, ...\}$

   Range of the relation $R = \{..., -6, -3, 0, 3, 6, 9, ...\}$.

   b. $R = \{..., (-2, -4), (-1, -2), (0,0), (1, 2), (2,4), (3, 6), ...\}$.

   Domain of the relation $R = \{..., -4, -2, 0, 2, 4, 6, ...\}$

   Range of the relation $R = \{...,2, 4, 0, -2, -4, -6, ...\}$.

   Domain and Range are the set of all integers.

6.
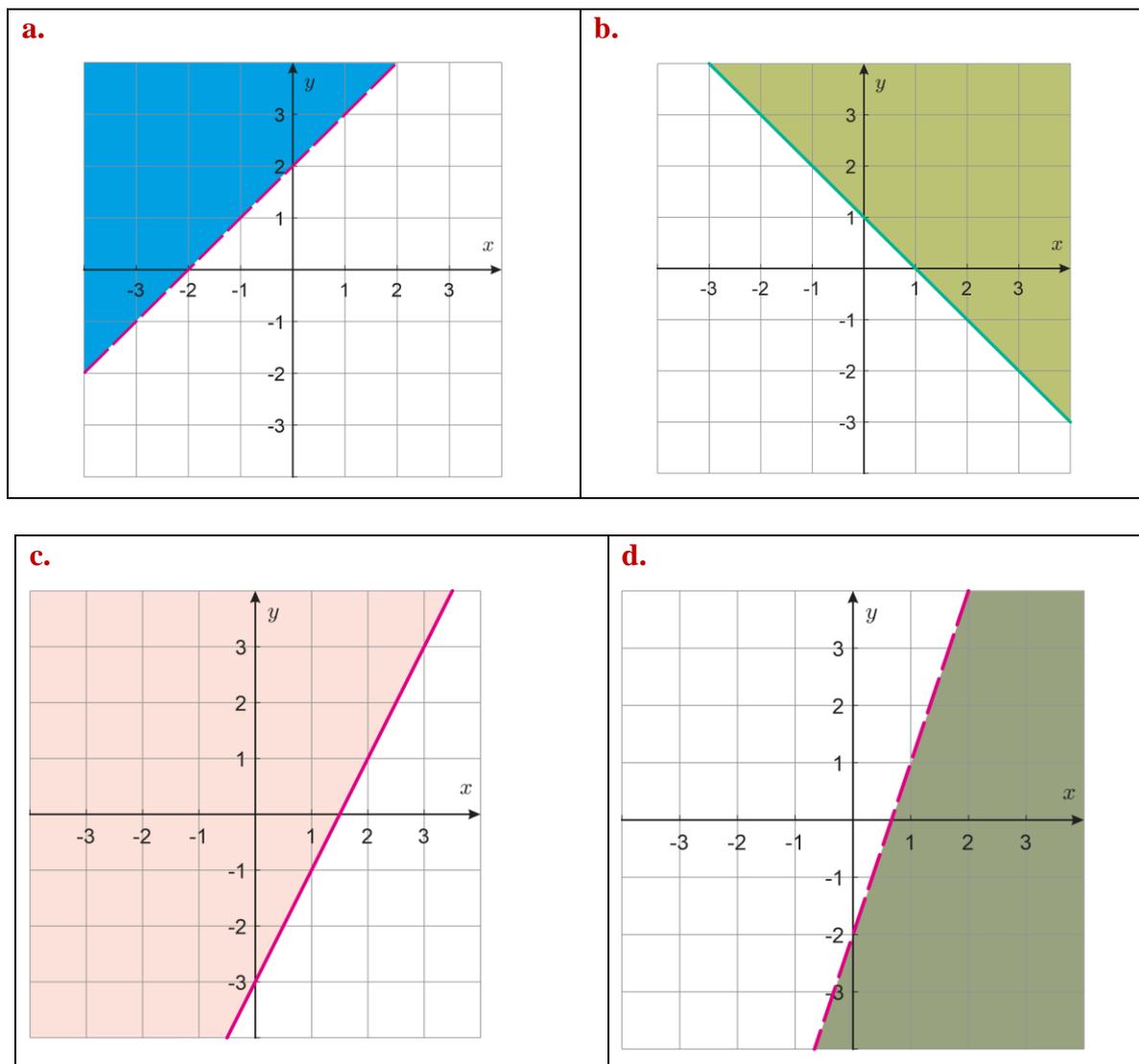
a.

b.

c.

d.

Self-Test Exercises 2

1. **a.** R is a function.

b. R is not a function. The same first element of R is mapped to distinct two second elements of R, that is 4 is mapped to 7 and 1.

c. R is not a function. For $x = 2$, multiples of x are $2, 4, 6, 8, 10, 12, ...$ that is, 2 is mapped to $2, 4, 6, 8, 10, 12, ...$

d. R is not a function. The same first element of R is mapped to distinct second element of R that is, $x = 4$ is mapped to -2 and 2.

2. Domain of F is a collection of $-2, 0, 4$ and 5 and the range of F is a collection of $1, 0, -2$ and -3.

3. Given: $f(x) = 3x + 3$ and $g(x) = x - 1$.

   a. $(f + g)(x) = f(x) + g(x) = (3x - 2) + (x + 1)$

$$= 3x - 2 + x + 1 = 4x - 1.$$

   $(f + g)(3) = 4 \times 3 - 1 = 11.$

  b. $(f - g)(x) = f(x) - g(x) = (3x - 2) - (x + 1) = 3x - 2 - x - 1 = 2x - 3.$

    $(f - g)(3) = 2 \times 3 - 3 = 3.$

4. **a.** $(f \cdot g)(x) = (-x + 1) \cdot x^2 = -x^3 + x^2$ and $(f \cdot g)(2) = -8 + 4 = -4.$

  b. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{-x+1}{x^2.}$ and $\left(\frac{f}{g}\right)(2) = -\frac{1}{4}.$

  c. $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{x^2}{-x+1.}$ and $\left(\frac{g}{f}\right)(2) = -4.$

Self-Test Exercises 3

1. The function $G1$ bijective, $G2$ is surjective, $F1$ is injective and $F2$ is bijective.

2. **a.** A is not one-to-one.   **b.** B is one-to-one.

3 **a.** onto   **b.** not onto

4. **a.** bijective   **b.** bijective

5. $f(x) = 4x + 1; \quad x = -3, -2, -1, 0, 1, 2, 3$

a.

x	−3	−2	−1	0	1	2	3
f(x) = 3x − 1	−10	−7	−4	1	2	5	8

b.

x	−1	−0.5	1	2	3	4
f(x) = −2x + 3	6	4	1	−1	−3	−5

c.

x	−1	0	1	2	3	4
f(x) = 4 − 2x	6	4	2	0	−2	−4

d.

x	−8	−4	−2	0	2	4
$f(x) = \frac{1}{2}x - 1$	−5	−3	−2	−1	0	1

6.

a. The graph of $f(x) = 1 - x^2$
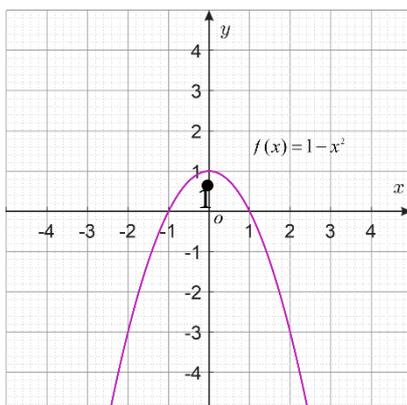
b. The graph of $g(x) = x^2 - 3$
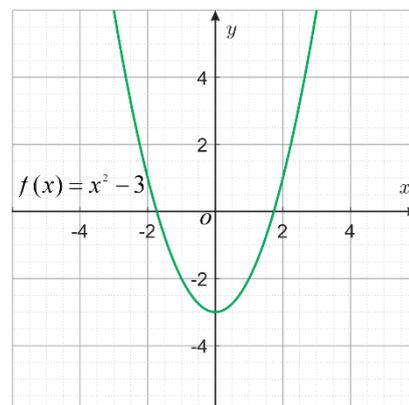
c. The graph of $h(x) = 2x^2 - 2$	**d.** The graph of $k(x) = (x - 2)^2 + 1$
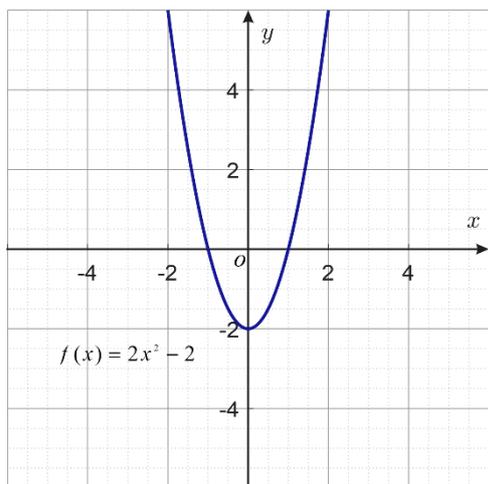
$f(x) = 2x^2 - 2$

$f(x) = (x-2)^2 + 1$
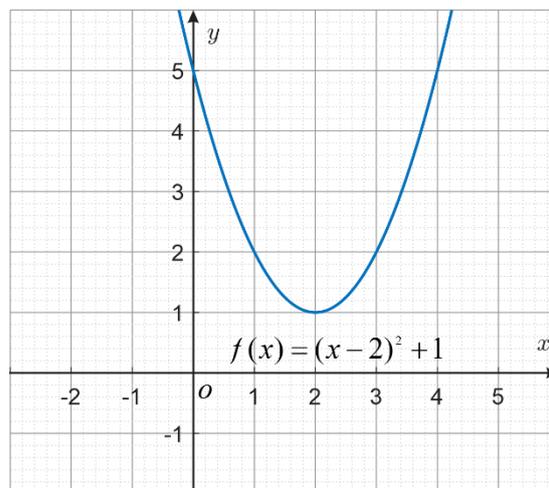
Self-Test Exercises 4

1. a. When the length is x m $(x \geq 0)$, the width (w) can be defined as follows:

$2x + 2w = 8$, $2w = 8 - 2x$, thus, $w = 4 - 2x$

b. $S = l \times w = x(4 - x)$

c. $f(x) = S = -x^2 + 4x = -(x - 2)^2 + 4$

The vertex is $(2, 4)$, y-intercept is 0.

The graph is shown on the right.

d. As $x \geq 0$ and $S \geq 0$,

Domain $= (0, 4)$, Range $= (0, 4)$

2. a. $y = S = \frac{1}{2}bh = \frac{1}{2} \times x \times 4 = 2x$

b. The graph is shown at the right.

c. Domain: $\{x | x \in \mathbb{R}\}$, Range: $\{y | y \in \mathbb{R}\}$.

$f(x)$

3. a. Vertex $= (-3, -1)$ and Axis of symmetry $x = -3$.

b. $f(x) = x^2 + 7x + 6 = \left(x + \frac{7}{2}\right)^2 + 6 - \frac{49}{4} = \left(x + \frac{7}{2}\right)^2 - \frac{25}{4}$.

Thus, the vertex is $(\frac{7}{2}, \frac{25}{4})$, and the axis of symmetry $x = \frac{7}{2}$.
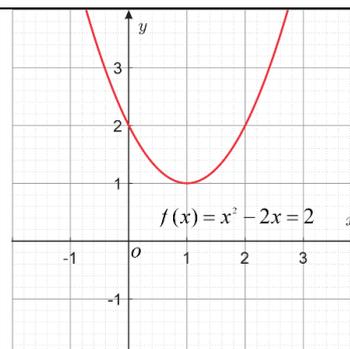
4.

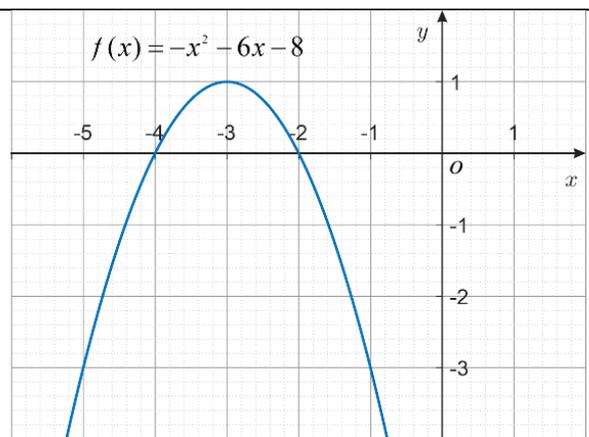|	
a. $f(x) = x^2 - 2x + 2 = (x - 1)^2 + 1$ The vertex is $(1, 1)$, and y-intercept is 2. The graph of the function opens upward, so the function has minimum value at vertex $(1, 1)$, which is 1.	 $f(x) = x^2 - 2x = 2$

b. $f(x) = -x^2 - 6x - 8 = -(x+3)^2 + 1$

The graph of the function opens downward, so the function has maximum value at vertex $(-3, 1)$, which is 1.

i.   Solution 1:

Let the length of one square and the other square be x and y respectively $(y > x)$. From the given condition,

$$\begin{cases} 4x + 4y = 40 \\ x^2 + y^2 = 58 \end{cases}$$

From the first equation, $x + y = 10$, then, $y = 10 - x$.

Substitute this to the second equation,

$$x^2 + (10 - x)^2 = 58$$
$$2x^2 - 20x + 42 = 0$$
$$x^2 - 10x + 21 = 0$$
$$(x - 3)(x - 7) = 0,$$

$x = 3, x = 7$, since $y > x$, then, $x = 3, y = 7$

Therefore, the length of each wire is $12\ cm$ and $28\ cm$.

Solution2:

Let the x and y be the larger and smaller lengths respectively.

$$x + y = 40 \quad . \quad . \quad . \quad (1)$$

After bending to form a square, area of each square is: $\left(\frac{x}{4}\right)^2$ and $\left(\frac{y}{4}\right)^2$ respectively.

$\frac{1}{16}(x^2 + y^2) = 58$. Which implies

$$x^2 + y^2 = 928 \ . \quad . \quad . \ (2)$$

Consider, $(x + y)^2 + (x - y)^2 = x^2 + y^2 + 2xy + x^2 + y^2 - 2xy$.

Therefore, $(x + y)^2 + (x - y)^2 = 2(x^2 + y^2)$. Using the values from equations (1) and (2)

$$(40)^2 + (x - y)^2 = 2(928)$$

$(x - y)^2 = 256 \Longrightarrow x - y = 16$. Since $x > y$, the difference $x - y$ cannot be negative.

$$x - y = 16. \quad . \quad . \ (3)$$

Solving simultaneous equations (1) and (3), $x = 28$ and $y = 12$
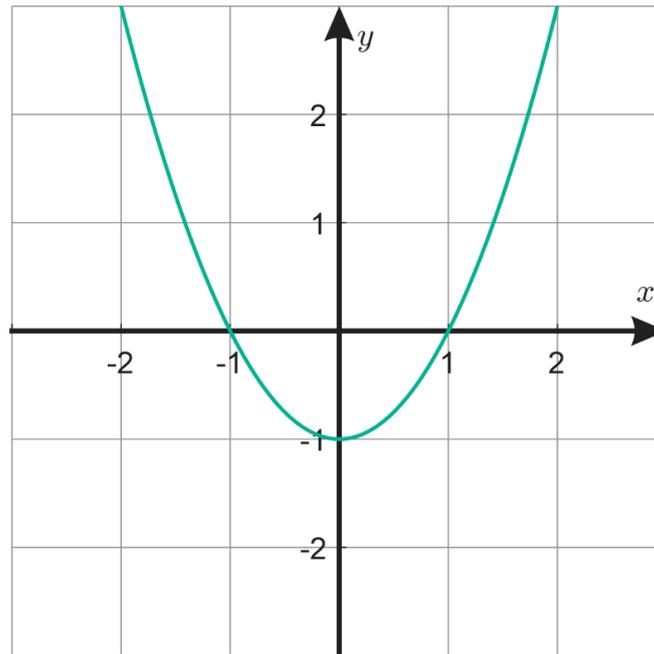
Lengths of pieces are 28 cm and 12 cm.

ii.

a. Draw the graph of the quadratic function.

$y = x^2 - 1$

x	...	-2	-1	0	1	2	...
y	...	3	0	-1	0	3	...

Read off the x-coordinate(s) of the point(s) where the curve crosses the x-axis.

The roots are $x = -1$ and $x = 1$. These are the solutions of $x^2 - 1 = 0$

b. The solution for the equation $x^2 + 2x + 1 = 0$ can be obtained by looking at the points where the graph $y = x^2 + 2x + 1$ cuts at the x-axis (i.e., $y = 0$).

$y = x^2 + 2x + 1 = (x + 1)^2$, hence the vertex is $(-1, 0)$

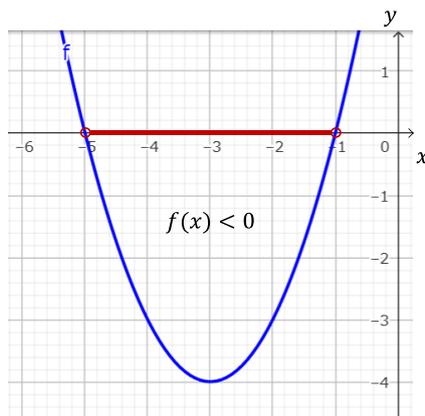| x	...	-3	-2	-1	0	1	...
y	...	4	1	0	1	4	...

Plot other points, y-intercept $(0, 1)$, and $(1, 4)$, $(-2, 1)$, (-3, 4) to draw graph. The graph $y = x^2 + 2x + 1 = 0$ cuts the $x - axis$ only at $x = -1$. So, the solution for the equation is $x = -1$.
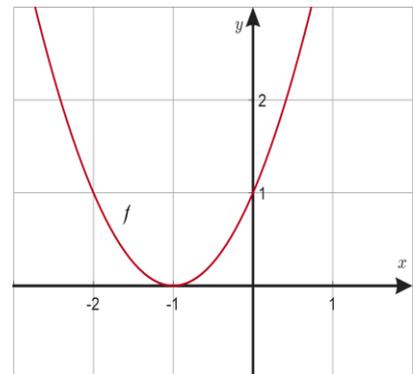
7. a. $x^2 + 6x + 5 < 0$

Draw the graph of $f(x) = x^2 + 6x + 5 = (x + 3)^2 - 4$

Plot the points: $(-5, 0), (-4, -3), (-3, -4), (-2, -3), (-1, 0)$.

From the graph, the value x that meets $f(x) < 0$ is $-5 < x < -1$. We also express the answer using the interval notation, $(-5, -1)$.

b. $x^2 + 6x - 7 \geq 0$

$f(x) = x^2 + 6x - 7 = (x + 3)^2 - 16$. The vertex is $(-3, -16)$. Using table to find other points.

x	-8	-7	-6	-5	0	1
$f(x) = x^2 - 6x - 7$	9	0	-7	-12	-7	0

The x-intercepts are $(-7, 0)$ and $(1, 0)$. We graph the x-intercepts, and the y-intercept. We connect these points to sketch the parabola.

Determine the solution from the graph.

The inequality asks for the values of x which make the function greater than equal to 0. Which values of x make the parabola above the x-axis? We include the values $= -7, 1$. The solution, in interval notation is $(-\infty, -7] \cup [1, \infty)$, or we simply express $x \leq -7, x \geq 1$.
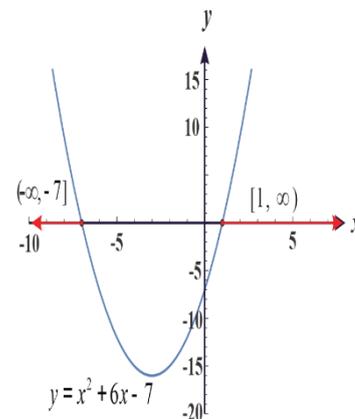
Unit 2

Self-Test Exercises 1

1. a. The degree is 2, the leading coefficient is -1 and the const. term is 10

 b. The degree is 3, the leading coefficient is 10 and the constant term is 2.

 c. The degree is -5, the leading coefficient is -5 and the constant term is -12.

 d. The degree is 4, the leading coefficient is 3 and the constant term is -5.

 e. The degree is 5, the leading coefficient is 10 and the constant term is -10.

2. a. Polynomial function with degree 4, leading coefficient 2 and constant term -6.

b. Polynomial function with degree 5, leading coefficient $\frac{3}{8}$ and constant term 7.

c. Not polynomial function because -2 and -1 are not positive integers.

d. Polynomial function with degree 1, leading coefficient $\sqrt{6}$ and constant term $\sqrt{2}$.

e. Polynomial function with degree 2, leading coefficient $-\frac{5}{3}$ and constant term $\frac{1}{2}$.

f. Not polynomial function because f can be rewritten as $f(x) = x^{-3} + x^{-4} + 1$ and -3 and -4 are not non-negative integers.

g. Polynomial function with degree 7, leading coefficient $\frac{8}{3}$ and constant term $\frac{5}{3}$.

h. Polynomial function with degree 3, leading coefficient 1 and constant term 2.

i. Polynomial function with degree 4, leading coefficient 1 and constant term 25.

j. Polynomial function with degree 3, leading coefficient 3 and constant term 8.

Self-Test Exercises 2

1. a. $f(x) + g(x) = (x^3 - 2x^2 - 3x + 4) + (-5x^4 + 3x^3 - 6x^2 + 2x + 3)$

$$= -5x^4 + (x^3 + 3x^3) + (-2x^2 - 6x^2) + (-3x + 2x) + (4 + 3);$$

grouping like terms

$$= -5x^4 + 4x^3 - 8x^2 - x + 7; \text{ adding like terms}$$

b. $f(x) + g(x) = (x^4 + 2x^3 - 5x^2 - 7x + 9) + (-5 + 7x + 8x^2 - 3x^3 - 2x^4)$

$$= (x^4 - 2x^4) + (2x^3 - 3x^3) + (-5x^2 + 8x^2) + (-7x + 7x) + (9 - 5);$$

grouping like terms

$$= -x^4 - x^3 + 3x^2 + 4; \text{ adding like terms}$$

c. $f(x) + g(x) = (6x^5 + x^4 - 2x^3 + 6x^2 + x - 5) + (4 + 2x - 3x^5 + 2x^4)$

$$= (6x^5 - 3x^5) + (x^4 + 2x^4) - 2x^3 + 6x^2 + (x + 2x) + (-5 + 4);$$

grouping like terms

$$= 3x^5 + 3x^4 - 2x^3 + 6x^2 + 3x - 1; \text{ adding like terms}$$

d. $f(x) + g(x) = \left(\sqrt{3}x^4 + 2x^3 - 5x^2 + x - 2\sqrt{5}\right) + (2\sqrt{5} - 3x - x^2 + 2x^3 + 3\sqrt{3}x^4)$

$$= \left(\sqrt{3}x^4 - 3\sqrt{3}x^4\right) + (2x^3 + 2x^3) + (-5x^2 - x^2) + (x - 3x) + (-\sqrt{5} + \sqrt{5});$$

grouping like terms

$$= -2\sqrt{3}x^4 + 4x^3 - 6x^2 - 2x; \text{ adding like terms}$$

2. a. $f(x) - g(x) = (-2x^3 + 5x^2 - 3x + 11) - (-9x^3 - x^2 + 6x - 3)$

$$= -2x^3 - 5x^2 - 3x + 11 + 9x^3 + x^2 - 6x + 3; \text{ removing brackets}$$

$$= (-2x^3 + 9x^3) + (-5x^2 + x^2) + (-3x - 6x) + (11 + 3); \text{Grouping like terms}$$

$$= 7x^3 - 4x^2 - 9x + 14; \text{ adding like terms}$$

b. $f(x) - g(x) = \left(-\frac{1}{5}x^4 + 2x^3 + 3x^2 - \frac{3}{4}x + 2\right) - (\frac{2}{5}x^4 - x^3 - 2x^2 + \frac{7}{4}x + 3)$

$= -\frac{1}{5}x^4 + 2x^3 + 3x^2 - \frac{3}{4}x + 2 - \frac{2}{5}x^4 + x^3 + 2x^2 - \frac{7}{4}x - 3$; removing brackets

$= \left(-\frac{1}{5}x^4 - \frac{2}{5}x^4\right) + (2x^3 + x^3) + (3x^2 + 2x^2) + (-\frac{3}{4}x - \frac{7}{4}x) + (2 - 3);$

Grouping like terms

$= -\frac{3}{5}x^2 + 3x^3 + 5x^2 - \frac{5}{2}x - 1$; adding like terms

c. $f(x) - g(x) = (-6x^5 + 2x^4 - 2x^3 + 2x^2 + 7x + 3) - (-2 + 5x - 6x^5 - 3x^4)$

$= -6x^5 + 2x^4 - 2x^3 + 2x^2 + 7x + 3 + 2 - 5x + 6x^5 + 3x^4$; removing brackets

$= (-6x^5 + 6x^5) + (2x^4 + 3x^4) - 2x^3 + 2x^2 + (7x - 5x) + (3 + 2);$

Grouping like terms

$= 5x^4 - 2x^3 + 2x^2 + 2x + 5$; adding like terms

d. $f(x) - g(x) = \left(3\sqrt{2}x^4 - 2x^3 + 5x^2 + x + 2\sqrt{7}\right) - \left(\sqrt{7} - 3x + x^2 - 2x^3 - 5\sqrt{2}x^4\right)$

$= 3\sqrt{2}x^4 - 2x^3 + 5x^2 + x + 2\sqrt{7} - \sqrt{7} + 3x - x^2 + 2x^3 + 5\sqrt{2}x^4$; removing brackets

$= \left(3\sqrt{2}x^4 + 5\sqrt{2}x^4\right) + (-2x^3 + 2x^3) + (5x^2 - x^2) + (x + 3x) + (2\sqrt{7} - \sqrt{7});$

Grouping like terms

$= 8\sqrt{2}x^4 + 4x^2 + 4x + \sqrt{7}$; adding like terms

3. a. $f(x).g(x) = (2x + 1)(4x^2 + x - 3)$

$= 2x(4x^2 + x - 3) + 1(4x^2 + x - 3)$

$= (8x^3 + 2x^2 - 6x) + (4x^2 + x - 3)$

$= 8x^3 + (2x^2 + 4x^2) + (-6x + x) - 3$

$= 8x^3 + 6x^2 - 5x - 3$

b. $f(x).g(x) = (-x^2 + 3x - 1)(3x^2 - x + 2)$

$= -x^2(3x^2 - x + 2) + 3x(3x^2 - x + 2) - 1(3x^2 - x + 2)$

$= (-3x^4 + x^3 - 2x^2) + (9x^3 - 3x^2 + 6x) + (-3x^2 + x - 2)$

$= -3x^4 + (x^3 + 9x^3) + (-2x^2 - 3x^2 - 3x^2) + (6x + x) - 2$

$= -3x^4 + 10x^3 - 8x^2 + 7x - 2$

c. $f(x).g(x) = (-x^3 + 3x)(x - 2x^2)$

$= -x^3(x - 2x^2) + (3x)(x - 2x^2)$

$= (-x^4 + 2x^5) + (3x^2 - 6x^3)$

$= -x^4 + 2x^5 + 3x^2 - 6x^3$

$= 2x^5 - x^4 - 6x^3 + 3x^2$

4. a.

$$\begin{array}{r} -x^2 + 3x - 2 \\ 4x + 3 \\ \hline -3x^2 + 9x - 6 \\ -4x^3 + 12x^2 - 8x \\ \hline -4x^3 + 9x^2 + x - 6 \end{array}$$

b.

$$2x^3 - x^2 - 5x + 1)$$
$$3x^2 + 2x)$$

$$\overline{4x^4 - 2x^3 - 10x^2 + 2x}$$

$$\frac{6x^5 - 3x^4 - 15x^3 + 3x^2}{6x^5 + x^4 - 17x^3 - 7x^2 + 2x}$$

c.

$$-x^3 + x^2 + 2x - 3)$$
$$2x^2 - 4x + 5)$$

$$\overline{-5x^3 + 5x^2 + 10x - 15}$$

$$4x^4 - 4x^3 - 8x^2 + 12x$$

$$\frac{-2x^5 + 2x^4 + 4x^3 - 6x^2}{-2x^5 + 6x^4 - 5x^3 - 9x^2 + 22x - 15}$$

5. a.

$$
\begin{array}{r}
2x^2 + 5x + 4 \\
x - 1 \overline{\smash{\big)}\, 2x^3 + 3x^2 - x - 1} \\
\underline{2x^3 - 2x^2} \\
5x^2 - x - 1 \\
\underline{5x^2 - 5x} \\
4x - 1 \\
\underline{4x - 4} \\
3
\end{array}
$$

Therefore, $q(x) = 2x^2 + 5x + 4$ and $r(x) = 3$

b.

$$
\begin{array}{r}
-3x^2 + x - 4 \\
-2x + 1 \overline{\smash{\big)}\, 6x^3 - 5x^2 + 9x + 2} \\
\underline{6x^3 - 3x^2} \\
-2x^2 + 9x + 2 \\
\underline{-2x^2 + x} \\
8x + 2 \\
\underline{8x - 4} \\
6
\end{array}
$$

Therefore, $q(x) = -3x^2 + x - 4$ and $r(x) = 6$

c.

$$
\begin{array}{r}
-x^2 + x + 2 \\
x - 2 \overline{\smash{\big)}\, -x^3 + 3x^2 + 8} \\
\underline{-x^3 + 2x^2} \\
x^2 + 8 \\
\underline{x^2 - 2x} \\
2x + 8 \\
\underline{2x - 4} \\
12
\end{array}
$$

Therefore, $q(x) = -x^2 + x + 2$ and $r(x) = 12$

Self-Test Exercises 3

1. a.

$$\begin{array}{r} -5x - 6 \\ -x+1 \enclose{longdiv}{5x^2 + x - 2} \\ 5x^2 - 5x \\ \hline 6x - 2 \\ 6x - 6 \\ \hline 4 \end{array}$$

$$5x^2 + x - 2 = (-x + 1)(-5x - 6) + 4$$

b.

$$\begin{array}{r} 2x + 7 \\ x^2 - 2x + 1 \enclose{longdiv}{2x^3 + 3x^2 - 8x - 4} \\ 2x^3 - 4x^2 + 2x \\ \hline 7x^2 - 10x - 4 \\ 7x^2 - 14x + 7 \\ \hline 4x - 11 \end{array}$$

$$2x^3 + 3x^2 - 8x - 4 = (x^2 - 2x + 1)(2x + 7) + (4x - 11)$$

c.

$$\begin{array}{r} 3x - 5 \\ x^3 + 1 \enclose{longdiv}{3x^4 - 5x^3 + 12x - 4} \\ 3x^4 + 3x \\ \hline -5x^3 + 9x - 4 \\ -5x^3 - 5 \\ \hline 9x + 1 \end{array}$$

$$3x^4 - 5x^3 + 12x - 4 = (x^3 + 1)(3x - 5) + (9x + 1)$$

d.

$$\begin{array}{r} -x^2 - 2x + 5 \\ x + 2 \enclose{longdiv}{-x^3 - 4x^2 + x + 2} \\ -x^3 - 2x^2 \\ \hline -2x^2 + x + 2 \\ -2x^2 - 4x \\ \hline 5x + 2 \\ 5x + 10 \\ \hline -8 \end{array}$$

$$-x^3 - 4x^2 + x + 2 = (x + 2)(-x^2 - 2x + 5) + (-8)$$

e.

$$\begin{array}{r}
-x^3 - 3x^2 - 9x \\
\hline
-x+3 \overline{\smash{\big)}\ \begin{aligned} x^4 \\ x^4 - 3x^3 \end{aligned}} \\
\end{array}$$

$$\begin{array}{r}
3x^2 \\
3x^3 - 9x^2 \\
\hline
9x^2 \\
9x^2 - 27x \\
\hline
27x \\
27x - 81 \\
\hline
81
\end{array}$$

$$x^4 = (-x+3)(-x^3 - 3x^2 - 9x) + 81$$

2. a. $c = 1$ and the remainder is $f(2) = 2(2)^3 - 3(2)^2 - 1 = 3$.

    b. $c = -1$ and the remainder is $f(-1) = -3(-1)^3 - 2(-1)^2 + (-1) - 5 = -5$.

    c. $c = 1$ and the remainder is $f(1) = 2(1)^{19} - 4(1)^{12} + 9(1) - 62 = 55$.

    d. $c = -1$ and the remainder is $f(-1) = (-1)^{16} + 8(-1)^3 + 99 = 92$.

    e. $c = \frac{1}{2}$ and the remainder is $f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + 3 = \frac{13}{4}$.

3. When $2x^4 - 2x^2 + bx - 4$ is divided by $x + 2$, the remainder is 5. Find the value of b

Let $f(x) = 2x^4 - 2x^2 + bx - 4$.

By the remainder theorem when $f(x)$ is divided by $x + 2 = x - (-2)$ the remainder is

$f(-2) = 2(-2)^4 - 2(-2)^2 + b(-2) - 4 = 20 - 2b$.

Since the remainder is 5, we have, $20 - 2b = 5$ and solving for b, we have, $b = 7.5$.

4. Let $f(x) = 5x^3 - ax^2 + bx - 1$.

Since $f(x)$ is divided by $x - 1$ the remainder is

$f(1) = 5(1)^3 - a(1)^2 + b(1) - 1 = -a + b + 4$.

Since the remainder is -2, we have, $-a + b + 4 = -2$

$$-a + b = -6 \ . \ . \ . \ (*)$$

Since $f(x)$ is divided by $x + 1$ the remainder is $f(-1) = 5(-1)^3 - a(-1)^2 + b(-1) - 1$

Since the remainder is 4, we have, $-5 - a - b - 1 = 4$

$$-a - b = 10 \ . \ . \ . \ (**)$$

Solving (*) and (**) simultaneously we get $a = -2$ and $b = -8$.

5. $x - 3$ has the form $x - c$, with the value of $c = 3$.

Now, $f(c) = f(3) = (3)^2 - 2(3) - 3 = 0$, then by the factor theorem $x - 3$ is a factor of $x^2 - 2x - 3$.

6. a. $c = 1$ and $f(1) = 1^3 - 2(1)^2 - 5 + 6 = 0$. Therefore, $x - 1$ is a factor.

    b. $c = 3$ and $f(3) = 3^3 - 2(3)^2 - 5(3) + 6 = 0$. Therefore, $x - 3$ is a factor.

   c. $c = -1$ and $f(-1) = (-1)^3 - 2(-1)^2 - 5(-1) + 6 = 8 \neq 0$. Therefore, $x + 1$ is not

      a factor

   d. $c = -2$ and $f(-2) = (-2)^3 - 2(-2)^2 - 5(-2) + 6 = 0$. Therefore, $x + 2$ is a factor

7. a. $f(1) = (1)^4 - (1)^3 - 13(1)^2 + (1) + 12 = 0$, $x - c = x - 1$ is a factor of $f(x)$.

  b. $f\left(-\frac{1}{2}\right) = 6\left(-\frac{1}{8}\right) - 5\left(\frac{1}{4}\right) - 12\left(-\frac{1}{2}\right) - 4 = 0$, $x - c = x + \frac{1}{2}$ is a factor of $f(x)$.

  c. $f(2) = 2^3 - 3(2^2) + 4(2) - 3 = 1 \neq 0$; $x - c = x - 2$ is not a factor of $f(x)$.

8.   **a.** Let $f(x) = 2kx^3 + x^2 + 5x + 1$, $x - 1$ is a factor of $f(x)$ implies

$$f(1) = 2k(1)^3 + (1)^2 + 5(1) + 2 = 0$$

$$2k = -8 \text{ and this implies } k = -4$$

  b. Let $f(x) = x^4 - 2kx^3 - 3kx^2 - x + 1$, $x + 2$ is a factor of $f(x)$ implies

$$f(-2) = (-2)^4 - 2k(-2)^3 - 3k(-2)^2 - (-2) + 1 = 0$$

$$4k + 19 = 0 \text{ and this implies } k = -\frac{19}{4}.$$

9.   Since $x - 1$ is a factor of $f(x)$ $x^4 + 2ax^3 - 3x^2 - 3bx + 8$

$$f(1) = (1)^4 + 2a(1)^3 - 3(1)^2 - 3b(1) + 8 = 0$$

$$2a - 3b = -6 \ . \ . \ . \ (*)$$

  Since $x + 2$ is a factor of $f(x)$

$$f(-2) = (-2)^4 + 2a(-2)^3 - 3(-2)^2 - 3b(-2) + 8 = 0$$

$$-16a + 6b = -12 \ . \ . \ . \ (**)$$

  Solving (*) and (**) simultaneously $a = 2$ and $b = \frac{10}{3}$.

Self-Test Exercises 4

1. a. $f(x) = (x - 2)(x + 4)(5x - 2)$

$$(x - 2)(x + 4)(5x - 2) = 0$$

$$x - 2 = 0, \quad x + 4 = 0 \text{ or } 5x - 2 = 0$$

$$x = 2, x = -4 \text{ or } x = \frac{2}{5}$$

      Therefore, the zeros of $f(x)$ are $2, -4$ and $\frac{2}{5}$.

  b. $f(x) = x^4 - 10x^2 + 9$

$$x^4 - 10x^2 + 9 = 0$$

$$(x^2)^2 - 10x^2 + 9 = 0, \text{ let } y = x^2$$

$$y^2 - 10y + 9 = 0$$

$$(y - 1)(y - 9) = 0$$

$$y - 1 = 0 \text{ or } y - 9 = 0$$

$$y = 1 \text{ or } y = 9$$

Substituting the value of y, we get $x^2 = 1$ or $x^2 = 9$

$x^2 = 1$ implies $x = -1$ or $x = 1$.

$x^2 = 9$ implies $x = -3$ or $x = 3$.

Therefore, the zeros of $f(x)$ are $-3, -1, 1$ and 3.

c. $f(x) = x^4 - x^2 - 6$

$x^4 - x^2 - 6 = 0$

$(x^2)^2 - x^2 - 6 = 0$, let $y = x^2$

$y^2 - y - 6 = 0$

$(y - 3)(y + 2) = 0$

$y - 3 = 0$ or $y + 2 = 0$

$y = 3$ or $y = -2$

Substituting the value of y, we get $x^2 = 3$ or $x^2 = -2$

$x^2 = 3$ implies $x = -\sqrt{3}$ or $x = \sqrt{3}$.

$x^2 = -1$ has no root in the set of real numbers (there is no real number whose square is a negative number)

Therefore, the zeros of $f(x)$ are $-\sqrt{3}$ and $\sqrt{3}$.

d. $f(x) = 2x^3 + 5x^2 - 3x$

$2x^3 + 5x^2 - 3x = 0$

$x(2x^2 + 5x - 3) = 0$

$x(2x^2 - x + 6x - 3) = 0$

$x(x(2x - 1) + 3(2x - 1)) = 0$

$x(2x - 1)(x + 3) = 0$

$x = 0, \ 2x - 1 = 0$ or $x + 3 = 0$

Therefore, the zeros of $f(x)$ are $0, \frac{1}{2}$ and -3.

2. a. $t = -\frac{2}{3}, t = 0$ **b.** $t = -\sqrt{2}, t = -2, t = -\frac{1}{3}$

3. If -1 and 2 are the zeros of $f(x)$, then $(x + 1)$ and $(x - 2)$ are factors of $f(x)$.

Therefore, $f(x) = k(x + 1)(x - 2)$ for some constant $k \neq 0$.

To find the value of k, we use the given condition $f(3) = 12$, that is,

$k(3 + 1)(3 - 2) = 4k = 12$. This implies $k = 3$.

Hence, the required polynomial function is,

$f(x) = 3(x + 1)(x - 2) = 3x^2 - 3x - 6$.

4. a. $f(1) = -1 < 0$ and $f(2) = 1 > 0$, by the location theorem there is a zero of $f(x)$ between 1 and 2.

b. $f(-1) = -3 < 0$ and $f(0) = 1 > 0$, by the location theorem there is a zero of $f(x)$

between -1 and -0.

5. a. $f(x) = 2x^2 + 7x + 3$

$f(x)$ has leading coefficient 2 and constant term 3.

Possible values of p are factors of 3. These are $\pm 1, \ \pm 3$.

Possible values of q are factors of 2. These are $\pm 1, \pm 2$.

The possible rational zeros $\frac{p}{q}$ are $\pm 1, \ \pm\frac{1}{2}, \ \pm 3$ and $\pm\frac{3}{2}$. Since $f(x)$ is a polynomial function

of degree 2, it has at most 2 zeros, and from the eight possible rational zeros at most 2 can be

the zeros of f. Since $f\left(-\frac{1}{2}\right) = 0$ and $f(-3) = 0$, the zeros of $f(x)$ are $-\frac{1}{2}$ and -3.

b. $f(x) = 6x^3 - 13x^2 + 4x + 3$

$f(x)$ has leading coefficient 6 and constant term 3.

Possible values of p are factors of 3. These are ± 1 and ± 3 .

Possible values of q are factors of 6. These are $\pm 1, \pm 2, \ \pm 3$ and ± 6.

The possible rational zeros $\frac{p}{q}$ are $\pm 1, \ \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm 3$ and $\pm\frac{3}{2}$. Checking shows that

$f\left(-\frac{1}{3}\right) = 0$, $f(1) = 0$ and $f\left(\frac{3}{2}\right) = 0$. So, the given polynomial has $-\frac{1}{3}, 1$ and $\frac{3}{2}$ as rational

zeros.

Self-Test Exercises 5

1. a. $f(x) = y = 2x^2 + 5x + 2 = (2x + 1)(x + 2)$. By making $f(x) = y = 0$ and solving the

equation $2x^2 + 5x + 2 = (2x + 1)(x + 2)$, we get $x = -\frac{1}{2}$ or $x = -2$. By making

$x = 0$ we have $y = 2$. Thus, $x = -\frac{1}{2}$ and $x = -2$ are the $x -$ intercepts and $y = 2$ is the

y-intercept.

b. $y = 2x^2 + 5x + 2 = 2\left(x^2 + \frac{5}{2}x + \frac{25}{16} - \frac{25}{16} + 1\right) = 2\left(x + \frac{5}{4}\right)^2 - \frac{9}{8} = -\frac{9}{8} + 2\left(x + \frac{5}{4}\right)^2$

Since $2\left(x + \frac{5}{4}\right)^2 \geq 0$ for all real numbers x, $f(x) \geq -\frac{9}{8}$ for all values of x and $-\frac{9}{8}$ is the

minimum value of f. This minimum value of f is attained when $x = -\frac{5}{4}$. **The point**

$(-\frac{5}{4}, -\frac{9}{8})$ **is called turning point or vertex of the graph of f.**

c.

x	-3	-2	$-5/4$	-1	-0.5	0	1
$y = f(x)$	5	0	$-9/8$	-1	0	2	9
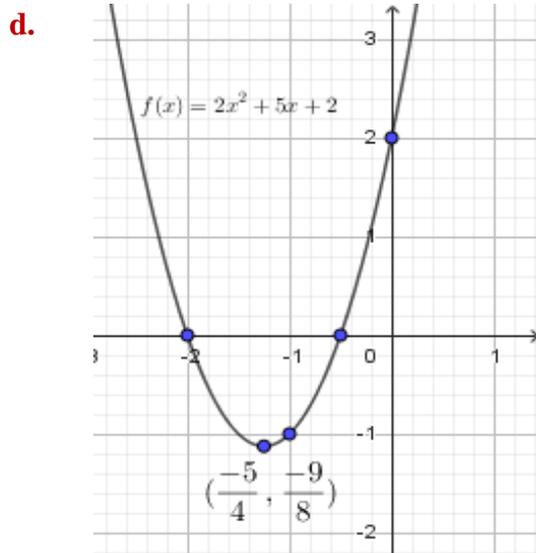
d.

$f(x) = 2x^2 + 5x + 2$

$\left(\dfrac{-5}{4}, \dfrac{-9}{8}\right)$

e. The domain is the set of all real numbers and the range is the set of all real numbers greater than or equal to $-\dfrac{9}{8}$.

2. a. $f(x) = y = -3x^2 + 4x = x(-3x + 4)$. By making $f(x) = 0$ and solving the

Equation it, we get $x = 0$ or $x = \dfrac{4}{3}$.

By making $x = 0$ we have $y = 0$. Thus, $x = 0$ and $x = \dfrac{4}{3}$ are the $x -$ intercepts

and $y = 0$ is the y-intercept.

b. $y = f(x) = -3x^2 + 4x = -3\left(x^2 - \dfrac{4}{3}x\right) = -3\left(x^2 - \dfrac{4}{3}x + \dfrac{4}{9}\right) + \dfrac{4}{3}$

$= \dfrac{4}{3} - 3\left(x - \dfrac{2}{3}\right)^2$

Since $3\left(x - \dfrac{2}{3}\right)^2 \geq 0$, $f(x) \leq \dfrac{4}{3}$ for all values of x and $\dfrac{4}{3}$ is the maximum value of f. This

maximum value of f is attained when $x = \dfrac{2}{3}$. The point $\left(\dfrac{2}{3}, \dfrac{4}{3}\right)$ is called turning point for the

graph of f.

c.

x	-0.5	0	$2/3$	1	$4/3$	2
$y = f(x)$	-2.75	0	$4/3$	1	0	-4

d.
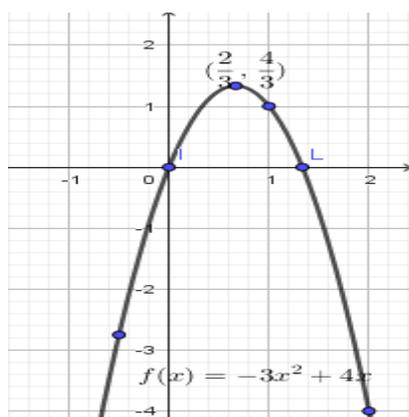
$\left(\dfrac{2}{3}, \dfrac{4}{3}\right)$

$f(x) = -3x^2 + 4x$

e. The domain is the set of all real numbers and the range is the set of all real numbers less than or equal to $\frac{4}{3}$.

3.   a. True   b. True   c. True   d. False   e. False   f. False

4.

a	i	Go upwards
	ii	Go upwards
	iii	Two
b	i	Go downwards
	ii	Go downwards
	iii	Three

c	i	Go downwards
	ii	Go upwards
	iii	Two
d	i	Go upwards
	ii	Go downwards
	iii	Two

5

	Leading Coefficient	**Degree**
a	Positive	odd
b	positive	even
c	negative	odd
d	Negative	even

Self-Test Exercises 6

1. a. The perimeter of the rectangular enclosure is $2x + y + (y - 4)$.

Since the length of the fencing wire is 80 meters

$$2x + y + (y - 4) = 80 \quad . \ . \ . \ (*)$$

Area of the rectangular garden is:
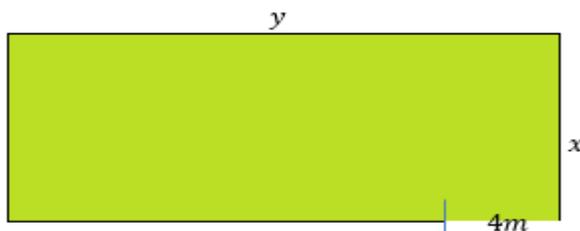
$$A = xy \quad . \ . \ . \ (**)$$

Solving for y from (*) and substituting the result in (**)

$$y = 42 - x$$
$$A(x) = 42x - x^2$$

b. $A(x) = 42x - x^2 = -x^2 + 42x = -(x^2 - 42x) = -(x^2 - 42x + (21)^2 - (21)^2)$

$$= -(x^2 - 42x + (21)^2) + (21)^2$$

$$= -(x - 21)^2 + (21)^2 = (21)^2 - (x - 21)^2$$

The area is maximum when $x = 21\ m$.

c. The maximum area is $(21)^2 = 441$ sq. m.

2. a. $V(x) = lwh = 2x(2x + 3)(3 - 2x) = 18x - 8x^3$.

b. $V(0.5) = 8\ \text{cm}^3$

c. No, because $V(2) = -28$ and volume cannot be negative number.

Unit 3

Self-Test Exercises 1

1. a. $3^2 = 3 \times 3 = 9$ **b.** $-3^2 = -(3 \times 3) = -9$ **c.** $(-3)^2 = -3 \times -3 = 9$

     d. $-(-4)^3 = -(-4 \times -4 \times -4) = -(-64) = 64$

2. a. $a^4 \times a^3 = a^{4+3} = a^7$ **b.** $\dfrac{a^4}{a^2} = a^{4-2} = a^2$ **c.** $(a^4)^2 = a^{4\times2} = a^8$

     d. $(2a)^3 = 2^3 a^3 = 8a^3$ **e.** $\left(\dfrac{a}{2}\right)^3 = \dfrac{a^3}{2^3} = \dfrac{a^3}{8}$

3. a) $3^0 = 1$ **b)** $(-10)^0 = 1$ **c)** $\left(\dfrac{24}{59}\right)^0 = 1$ **d)** $2^{-2} = \dfrac{1}{2^2} = \dfrac{1}{4}$

     e) $10^{-2} = \dfrac{1}{10^2} = \dfrac{1}{100}$ **f)** $\left(\dfrac{2}{3}\right)^{-3} = \dfrac{1}{\left(\frac{2}{3}\right)^3} = \dfrac{1}{\left(\frac{8}{27}\right)} = 1 \div \dfrac{27}{8} = 1 \times \dfrac{27}{8} = \dfrac{27}{8}$

     g) $\left(\dfrac{1}{2}\right)^{-5} = \dfrac{1}{\left(\frac{1}{2}\right)^5} = \dfrac{1}{\left(\frac{1}{32}\right)} = 1 \div \dfrac{1}{32} = 1 \times 32 = 32$

4.   **a)** $x^{-2} \times x^3 = x^{-2+3} = x^1 = x$

     b) $\dfrac{2^4}{2^7} = 2^{4-7} = 2^{-3} = \dfrac{1}{8}$

     c) $\dfrac{3^{-5}}{3^{-2}} = 3^{-5+2} = 3^{-3} = \dfrac{1}{3^3} = \dfrac{1}{27}$

     d) $(-x \times 2y)^2 = (-x)^2 \times (2y)^2 = (-1)^2 x^2 \times 2^2 y^2 = 4x^2 y^2$

     e) $n^{-3} \times n^{-2} = n^{-3+(-2)} = n^{-5} = \dfrac{1}{n^5}$

     f) $(4m)^3 \times (8m)^{-2} = (4^2 m^3) \times (8^{-3} m^{-2}) = (2^4 \times 2^{-9})(m^3 \times m^{-2}) = 2^{-5} m^1 = \dfrac{1}{2^5} m = \dfrac{1}{32} m$

     g) $2^t \times 2^{3t} \times 2^{2t} = (2^t \times 2^{3t}) \times 2^{2t} = 2^{t+3t} \times 2^{2t} = 2^{t+3t+2t} = 2^{6t}$

     h) $(2^3)^{2n} = 2^{3\times2n} = 2^{6n} = 64^n$

     i) $(2a^{-3} \times b^2)^{-2} = (2a^{-3})^{-2} \times (b^2)^{-2} = 2^{-2} \times a^{-3\times-2} \times b^{2\times-2} = \dfrac{1}{4} a^6 b^{-4}$

     j) $\dfrac{(a^2)^{-2} \times (a^3)^3}{a^9} = \dfrac{a^{-4} \times a^9}{a^9} = \dfrac{a^{-4+9}}{a^9} = \dfrac{a^5}{a^9} = \dfrac{1}{a^{9-5}} = \dfrac{1}{a^4}$

k) $\left(\frac{m^{-5}n^2}{n^{-2}m^6}\right)^{-1} = \frac{m^5n^{-2}}{n^2m^{-6}} = \frac{m^5}{m^{-6}} \times \frac{n^{-2}}{n^2} = \frac{m^{11}}{n^4}$

5. a) Since $3^4 = 81$, $\sqrt[4]{81} = (81)^{\frac{1}{4}} = 3$　　　　　　**b)** Since $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$, $\sqrt[5]{\frac{1}{32}} = \left(\frac{1}{32}\right)^{\frac{1}{5}} = \frac{1}{2}$

c) Since $(-5)^3 = -125$, $\sqrt[3]{-125} = (-125)^{\frac{1}{3}} = -5$　　**d)** Since $3^3 = 27$, $\sqrt[3]{27} = (27)^{\frac{1}{3}} = 3$

e) Since $(-10)^3 = -1000$, $-\sqrt[3]{-1000} = -(-1000)^{\frac{1}{3}} = -(-10) = 10$

f) $\sqrt[4]{-10000} = (-10000)^{\frac{1}{4}}$ is not a real number because there is no real number a such that a^4 is -10000

6. a) $(-27)^{-\frac{2}{3}} = ((-3)^3)^{-\frac{2}{3}} = (-3)^{3 \times -\frac{2}{3}} = (-3)^{-2} = \frac{1}{(-3)^2} = \frac{1}{9}$

b) $3^{\frac{4}{3}} \times 9^{\frac{4}{3}} = 3^{\frac{4}{3}} \times (3^2)^{\frac{4}{3}} = 3^{\frac{4}{3}} \times 3^{\frac{8}{3}} = 3^{\frac{4}{3}+\frac{8}{3}} = 3^{\frac{12}{3}} = 3^4 = 81$

c) $\frac{81^{\frac{2}{5}}}{27^{\frac{6}{5}}} = \frac{(3^4)^{\left(\frac{2}{5}\right)}}{(3^3)^{\left(\frac{6}{5}\right)}} = \frac{3^{\left(\frac{8}{5}\right)}}{3^{\left(\frac{18}{5}\right)}} = \frac{1}{3^{\left(\frac{10}{5}\right)}} = \frac{1}{3^2} = \frac{1}{9}$

d) $\left(m^{\frac{1}{4}} \times 3n^{\frac{5}{2}}\right)^2 = \left(m^{\frac{1}{4}}\right)^2 \times \left(3n^{\frac{5}{2}}\right)^2 = m^{\frac{1}{4}\times 2} \times 3^2 n^{\frac{5}{2}\times 2} = m^{\frac{1}{2}} \times 9n^{10} = 9m^{\frac{1}{2}}n^5$

e) $\left(\frac{a^{\frac{1}{3}}}{b^{-\frac{3}{2}}}\right)^{-6} = \frac{a^{\left(\frac{1}{3}\right)(-6)}}{b^{\left(-\frac{3}{2}\right)(-6)}} = \frac{a^{-3}}{b^9} = \frac{1}{a^3b^9}$　　　　**f)** $\frac{(x^2)^{-\frac{1}{4}} \times (x^3)^{\frac{2}{9}}}{x^{\frac{1}{2}}} = \frac{x^{-\frac{1}{2}} \times x^{\frac{2}{3}}}{x^{\frac{1}{2}}} = \frac{x^{\frac{1}{6}}}{x^{\frac{1}{2}}} = \frac{1}{x^{\frac{1}{3}}}$

g) $\left(\frac{x^{\frac{1}{8}} y^{-\frac{1}{3}}}{y^{\frac{2}{3}} x^{\frac{1}{4}}}\right)^4 = \frac{1}{x^{\frac{1}{2}}y^4} = \frac{1}{\sqrt{x}y^4}$

7. a) $\sqrt[5]{81} = 81^{\frac{1}{5}} = 3^{4\times\frac{1}{5}} = 3^{\frac{4}{5}}$　　**b)** $\sqrt[3]{32} = 32^{\frac{1}{3}} = 2^{5\times\frac{1}{3}} = 2^{\frac{5}{3}}$　　**c)** $\frac{\sqrt[4]{25}}{\sqrt[4]{5}} = \frac{25^{\frac{1}{4}}}{5^{\frac{1}{4}}} = \left(\frac{25}{5}\right)^{\frac{1}{4}} = 5^{\frac{1}{4}}$

d) $\frac{\sqrt[3]{40}}{\sqrt[3]{5}} = \frac{40^{\frac{1}{3}}}{5^{\frac{1}{3}}} = \left(\frac{40}{5}\right)^{\frac{1}{3}} = 8^{\frac{1}{3}} = 2^{\frac{3}{3}} = 2$　　**e)** $\left(\sqrt[4]{27}\right)^6 = \left(27^{\frac{1}{4}}\right)^6 = \left(3^{\frac{3}{4}}\right)^6 = 3^{\frac{9}{2}}$

f) $\left(\sqrt[3]{121}\right)^2 = \left(121^{\frac{1}{3}}\right)^2 = \left(11^{\frac{2}{3}}\right)^2 = 11^{\frac{4}{3}}$

8. a) $3^{\sqrt{3}} \times 3^{\sqrt{3}} = 3^{2\sqrt{3}}$　　　　　　　　　**b)** $\left(4^{\sqrt{2}}\right)^2 = \left(2^{2\sqrt{2}}\right)^2 = 2^{4\sqrt{2}}$　　**c)** $\left(\sqrt[3]{8}\right)^{-2} = \frac{1}{4}$

d) $\sqrt{2}^{\sqrt{3}} \times \sqrt{2}^{\sqrt{27}} = \sqrt{2}^{3\sqrt{3}}$　　　　　　**e)** $\frac{3^{\sqrt{2}+3}}{3^{\sqrt{2}-1}} = 3^{(\sqrt{2}+3)-(\sqrt{2}+3)} = 3^4 = 81$

f) $\left(2^{\sqrt{3}}\right)^{\sqrt{27}} = 2^9 = 512$　　　　　　　　**g)** $\frac{3^{\sqrt{2}} \times 9^{\sqrt{8}}}{27^{\sqrt{18}}} = \frac{1}{3^{4\sqrt{2}}} = \frac{1}{81^{\sqrt{2}}}$

h) $\frac{(5^{\sqrt{3}})^2 \times 5^{-\sqrt{12}} \times 25^{\sqrt{3}}}{5^{\sqrt{27}}} = \frac{1}{5^{\sqrt{3}}}$

Self-Test Exercises 2

1. **a)** $2^8 = 256$　if and only if　$\log_2 256 = 8$

b) $2^{-7} = \frac{1}{128}$　if and only if　$\log_2 \frac{1}{128} = -7$

c) $\sqrt[4]{625} = 5$　if and only if　$\log_{625} 5 = \frac{1}{4}$

d) $81^{-\frac{1}{2}} = \frac{1}{9}$ if and only if $\log_{81} \frac{1}{9} = -\frac{1}{2}$

2. **a)** $\log_{10} 10000 = 4$ if and only if $10^4 = 10,000$

b) $\log_8 8 = 1$ if and only if $8^1 = 8$

c) $\log_{10} 0.0001 = -4$ if and only if $10^{-4} = 0.0001$

d) $\log_3 \frac{1}{81} = -4$ if and only if $3^{-4} = \frac{1}{81}$

3. **a)** $\log_9 81$

Let $\log_9 81 = x$. Then $9^x = 81 = 9^2$ i.e., $9^x = 9^2$ if and only if $x = 2$.

So, $\log_9 81 = 2$.

b) $\log_8 32$

Let $\log_8 32 = y$. Then $8^y = 32$ i.e., $2^{3y} = 2^5$ if and only if $y = \frac{5}{3}$.

So, $\log_8 32 = \frac{5}{3}$.

c) $\log_{1000} 0.0001$

Put $t = \log_{1000} 0.0001$ so that $1000^t = 0.0001 = \frac{1}{10000} = \frac{1}{10^4} = 10^{-4}$ that is.

$1000^t = 10^{3t} = 10^{-4}$ if and only if $3t = -4$ which implies $t = -\frac{4}{3}$.

Hence, $\log_{1000} 0.0001 = -\frac{4}{3}$.

d) $\log_{\sqrt{8}} 2$

Let $\log_{\sqrt{8}} 2 = m$. Then $\sqrt{8}^m = 2$

But, $\sqrt{8}^m = \left[(8)^{\frac{1}{2}}\right]^m = \left[(2^3)^{\frac{1}{2}}\right]^m = 2^{\frac{3}{2}m}$

So, $2^{\frac{3}{2}m} = 2^1$ if and only if $m = \frac{2}{3}$

Consequently, $\log_{\sqrt{8}} 2 = \frac{2}{3}$

4. **a)** 3 **b)** $2p$ **c)** 1 **d)** $2x$

5. **a)** $\log_5 \sqrt[3]{5} = \log_5 5^{\frac{1}{3}} = \frac{1}{3}\log_5 5 = \frac{1}{3}$ **b)** $\log_7 49 = 2$

c) $\log_{10}\left(\frac{1}{10000}\right) = \log_{10} 10^{-4} = -4$ **d)** $\log_{\left(\frac{1}{0.1}\right)}(100) = \log_{10}(10)^2 = 2$

e) $\log_2 \sqrt[3]{\frac{1}{32}} = \log_2 2^{-\frac{5}{3}} = -\frac{5}{3}$ **f)** $\log_5 100 + \log_5 0.25 = \log_5(100 \times 0.25) = \log_5 25 = 2$

6. **a)** $\log_3 162 - \log_3 2 = \log_3\left(\frac{162}{2}\right) = \log_3 81 = \log_3 3^4 = 4\log_3 3 = 4 \times 1 = 4$

b) $\log_2 3 - \log_2 24 = \log_2\left(\frac{3}{24}\right) = \log_2\left(\frac{1}{8}\right) = -3 \times \log_2 2 = -3 \times 1 = -3$

c) $\log_3 \sqrt{486} - \log_3 \sqrt{2} = \log_3\left(\frac{\sqrt{486}}{\sqrt{2}}\right) = \log_3 \sqrt{243} = \frac{5}{2}$

d) $\log_6 9 + \log_6 120 - \log_6 5 = (\log_6 9 + \log_6 120) - \log_6 5 = \log_6 \frac{9 \times 120}{5} = \log_6 216 = 3$

e) $\log_5 3 - \log_5 12 + \log_5 20 = (\log_5 3 - \log_5 12) + \log_5 20$

$$= \log_5\left(\frac{3}{12} \times 20\right) = \log_5 5 = 1$$

f) $\log_{20} 120 - 2\log_{20} 3 + \log_{20} 30 = 2$

7. a) $\log_{\sqrt{5}} 125 = \frac{\log_5 125}{\log_5 \sqrt{5}} = \frac{3\log_5 5}{\frac{1}{2}\log_5 5} = 6$ **b)** $\log_{\sqrt{7}} 49 = \frac{\log_7 49}{\log_7 \sqrt{7}} = \frac{2\log_7 7}{\frac{1}{2}\log_7 7} = 4$

c) $\log_{\left(\frac{1}{4}\right)} 256 = \frac{\log_2 256}{\log_2\left(\frac{1}{4}\right)} = \frac{8\log_2 2}{-2\log_2 2} = -4$ **d)** $\log_{36}\left(\frac{1}{6}\right) = \frac{\log_6 \frac{1}{6}}{\log_6 36} = \frac{-\log_6 6}{2\log_6 6} = -\frac{1}{2}$

e) $\log_{0.001} 10 = \frac{\log_{10} 10}{\log_{10} 0.001} = \frac{\log_{10} 10}{-3\log_{10} 10} = -\frac{1}{3}$

8. a) $\log\left(\frac{\sqrt{1000}}{0.0001}\right) = \log\sqrt{1000} - \log 0.0001 = \frac{3}{2} - 4 = -\frac{5}{2}$

   b) $\log 100\sqrt{0.01} = 2\log_{10} 10 - \log_{10} 10 = 1$

   c) $\log \sqrt[4]{1000} = \log 10^{\frac{3}{4}} = \frac{3}{4}$

9. a) Reading the number in row 3.5 under column 2, gives 0.5465; $\log 3.52 = 0.5465$.

   b) Reading the number in row 1.9 under column 8, gives 0.2967; $\log 1.98 = 0.2967$.

   c) Reading the number in row 5.4 under column 0, gives 0.7324; $\log 5.4 = 0.7324$.

   d) Read the number at the intersection of row 7.2 and column 5, this gives 0.8603.

   From the mean difference part of the common logarithm table, read the number

   at the intersection of row 7.2 and column 4, this gives 0.0002.

   Adding the two results gives 0.8605; $\log 7.254 = 0.8605$.

   e) Since, $0.25 = 2.5 \times 10^{-1}$, we have

   $\log 0.25 = \log(2.50 \times 10^{-1}) = \log 2.50 + \log 10^{-1} = \log 2.50 - 1$.

   Reading the number in row 2.5 under column 0, gives 0.3979; $\log 2.5 = 0.3979$.

   So, $\log 0.25 = \log 2.50 - 1 = 0.3979 - 1 = -0.6021$.

   f) Reading the number in row 8.9 under column 9, gives 0.9538 ; $\log 8.99 = 0.9538$

   g) Since, $0.0007 = 7.00 \times 10^{-4}$, we have

   $\log 0.0007 = \log(7.00 \times 10^{-4}) = \log 7.00 + \log 10^{-4} = \log 7.00 - 4$

   Reading the number in row 7.0 under column 0, gives 0.8451 ; $\log 7.00 = 0.8451$

   So, $\log 0.0007 = \log 7.00 - 4 = 0.8451 - 4 = -3.1549$.

   h) Since, $500 = 5.00 \times 10^2$, we have

   $\log 500 = \log(5.00 \times 10^2) = \log 5.00 + \log 10^2 = \log 5.00 + 2$.

   Reading the number in row 5.0 under column 0, gives 0.6990 ; $\log 5.00 = 0.6990$.

   So, $\log 500 = \log 5.00 + 2 = 0.6990 + 2 = 2.6990$.

10. a) $0.00072 = 7.20 \times 10^{-4}$. So, the characterstic is -4 and the mantisa is $\log 7.20$.

   b) $0.46 = 4.6 \times 10^{-1}$; the characterstic is -1 and the mantisa is $\log 4.6$.

   c) $603 = 6.03 \times 10^2$. So, the characterstic is 2 and the mantisa is $\log 6.03$.

d) $\frac{1}{4} = 0.25 = 2.5 \times 10^{-1}$; the characterstic is -1 and the mantisa is $\log 2.5$.

e) $5.5 = 5.5 \times 10^0$. Therefore, the characterstic is 0 and the mantisa is $\log 5.5$.

f) $7 = 7 \times 10^0$; the characterstic is 0 and the mantisa is $\log 7$.

g) $5420 = 5.42 \times 10^3$; the characterstic is 3 and the mantisa is $\log 5.42$.

h) $65.615 = 6.5615 \times 10^1$; the characterstic is 1 and the mantisa is $\log 6.5615$.

11. a) Reading the at the intersection of row 0.63 and Column 2 from the antilogarithm table gives 4.285. From the mean difference part of the antilogarithm table read the number at the intersection of row 0.63 and column 1 gives 0.001. Adding the two results gives 4.286. So, the antilog $0.6321 = 4.286$.

b) Reading the number at the intersection of row 0.84 and Column 4 from the antilogarithm table gives 6.982. So, antilog $0.8440 = 6.982$

c) antilog $0.7779 = 5.984 + 0.012 = 5.996$.

d) antilog $0.8 = 6.310$.

e) antilog $1.4050 = $ antilog$(0.4050 + 1) = $ antilog$(0.4050) \times 10^1 = 2.541 \times 10^1 = 25.41$.

f) antilog $0.8574 = 7.194 + 0.007 = 7.201$

g) antilog $6.645 = $ antilog$(.645 + 6) = $ antilog$(0.645) \times 10^6 = 4.416 \times 10^6 = 4{,}416{,}000$

h) antilog $(-0.3) = $ antilog$\big((1 - 0.3) - 1\big) = $ antilog$(0.7 - 1) = 5.012 \times 10^{-1} = 0.5012$

12. a) Let $x = 3.34 \times 7.45$.

Then $\log x = \log(3.34 \times 7.45) = \log 3.34 + \log 7.45 = 0.5237 + 0.8722 = 1.3959$

So, $x = $ antilog$(1.3959) = $ antilog$(0.3959 + 1) = 2.488 \times 10^1 = 24.88$

Therefore, $3.34 \times 7.45 = 24.88$

b) Let $x = \sqrt[4]{20} = 20^{\frac{1}{4}}$.

Then $\log x = \log 20^{\frac{1}{4}} = \frac{1}{4}(\log 20) = \frac{1}{4}(\log 2 \times 10^1) = \frac{1}{4}(0.301 + 1)$

$\qquad\qquad = \frac{1}{4}(1.301) = 0.3252$.

Hence, $x = $ antilog$(0.3252) = 2.114$.

Therefore, $\sqrt[5]{25} \approx 2.114$.

c) Let $x = 4^{1.53}$.

Then $\log x = \log 4^{1.53} = 1.53 \log 4 = 1.53(0.6021) = 0.9212$.

$x = $ antilog$(0.9212) = 8.341$.

So, $4^{1.53} \approx 8.341$.

d) Let $x = (5.4)^{2.3} \times (0.33)^{5.2}$.

Then $\log x = \log\big((5.4)^{2.3} \times (0.33)^{5.2}\big) = \log(5.4)^{2.3} + \log(0.33)^{5.2}$

$\qquad\qquad = 2.3 \log 5.4 + 5.2 \log 0.33 = 2.3 \times (0.7324) + 5.2 \times (0.5185 - 1)$

$$= -0.8193 = 1 - 0.8193 - 1 = 0.1807 - 1.$$

So, $x = \text{antilog}(0.1807 - 1) = 1.517 \times 10^{-1} = 0.1517.$

Therefore, $(5.4)^{2.3} \times (0.33)^{5.2} \approx 0.1517.$

e) Let $x = \dfrac{\sqrt{299}}{(3.24)^2}.$

Then $\log x = \log \dfrac{\sqrt{299}}{(3.24)^2}$

$$= \log \sqrt{299} - \log(3.24)^2 = \log 299^{\frac{1}{2}} - \log(3.24)^2 = \frac{1}{2}(\log 2.99 \times 10^2) - 2(\log 3.24)$$

$$= \frac{1}{2}[0.4757 + 2] - 2[0.5105] = 0.2168.$$

That is, $\log x = 0.2168$ if and only if $x = \text{antilog}(0.2168) = 1.647.$

Thus, $\dfrac{\sqrt{299}}{(3.24)^2} \approx 1.647.$

Self-Test Exercises 3

1. a) 1   **b)** 8   **c)** $\dfrac{1}{2}$ **d)** 2

2. a) $f(x) = 5^{2x}$ **b)** $f(x) = 5^{\frac{1}{3}x}$ **c)** $f(x) = 5^{-x}$ **d)** $f(x) = 5^{\frac{4}{5}x}$ **e)** $f(x) = 5^{-x}$ \

3. a) Complete the table of values below

x	-1	0	1
$y = f(x)$	0.2	1	5

  b) y-intercept $= 1$ and has no x-intercept

  c)
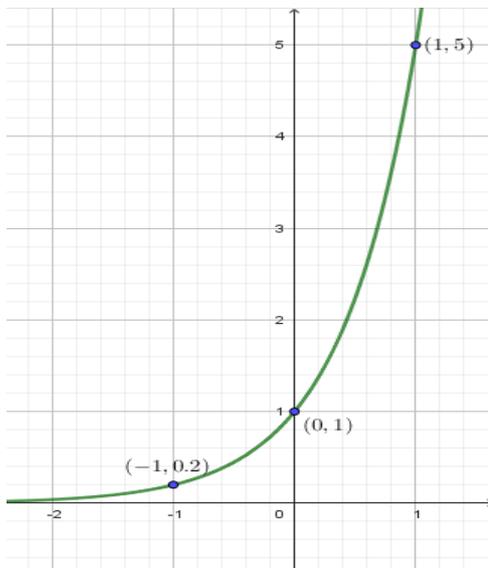
  d) Domain is the set of real numbers and the range is the set of positive real numbers.

4 a) Complete the table of values below

x	-1	0	1
$y = f(x)$	5	1	0.2

b) y-intercept $= 1$ and has no x-intercept

c)

$(-1, 5)$

$(0, 1)$

$(1, 0.2)$
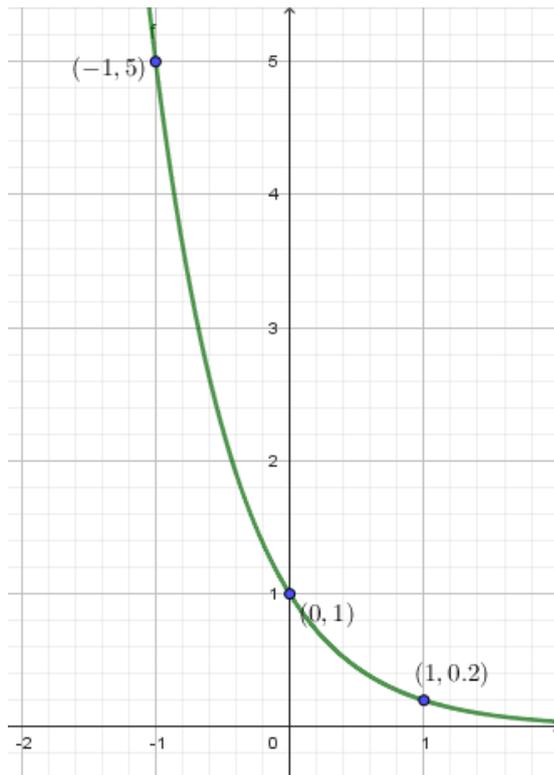
d) Domain is the set of real numbers and the range is the set of positive real numbers.

5.

$(-1, 5)$

$(1, 5)$

$(0, 1)$

$(-1, 0.2)$

$(1, 0.2)$
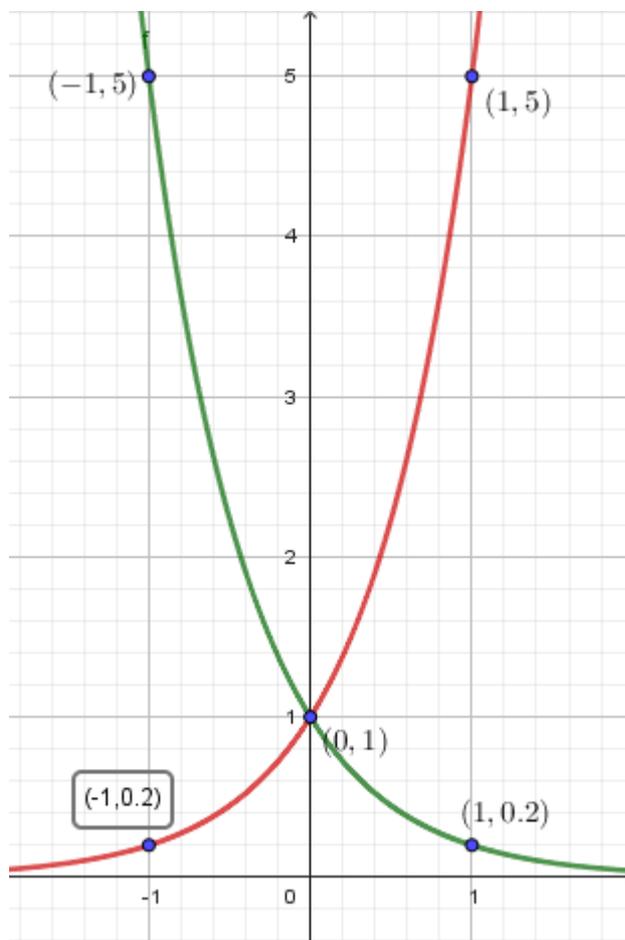
Self-Test Exercises 4

1. a) $-\dfrac{1}{2}$ **b)** Does not exist   **c)** $\dfrac{1}{2}$ **d)** $-\dfrac{1}{4}$

2. a) $f(x) = \dfrac{1}{2}\log_5 x$ **b)** $f(x) = \dfrac{2}{3}\log_5 x$ **c)** $f(x) = -\log_5 x$ **d)** $f(x) = 3\log_5 x$

3. a)

x	$\dfrac{1}{125}$	$\dfrac{1}{25}$	$\dfrac{1}{5}$	1	5
$f(x) = \log_5 x$	-3	-2	-1	0	1

  b) Has no y-intercept and the x-intercept is $(1,0)$.

  c)
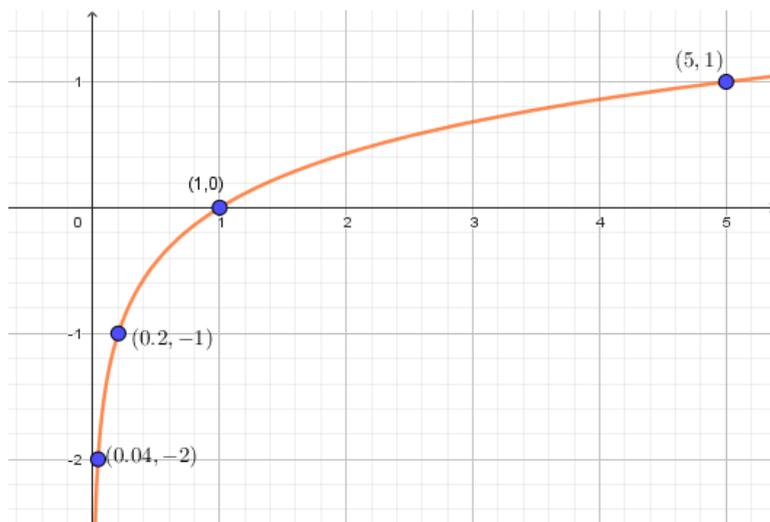
  d) Domain $= (0, \infty)$ and Range is the set of all real numbers.

4. a)

x	$\dfrac{1}{125}$	$\dfrac{1}{25}$	$\dfrac{1}{5}$	1	5
$f(x) = \log_{\frac{1}{3}} x$	3	2	1	0	-1

  b) Has no y-intercept and the x-intercept is $(1,0)$.

  c)
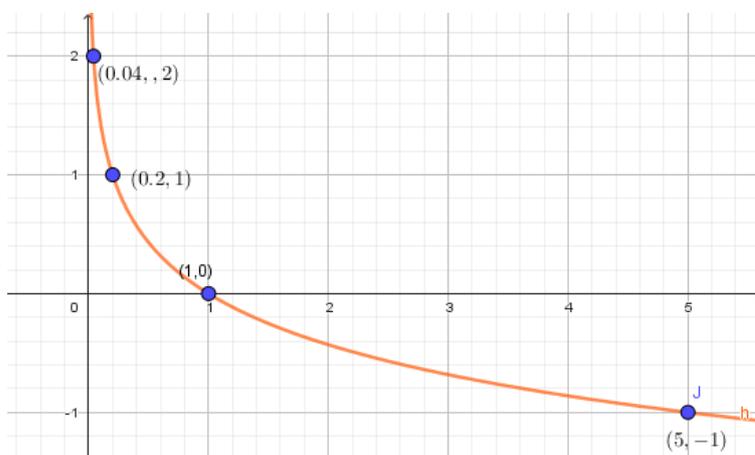
d) Domain $= (0, \infty)$ and Range is the set of all real numbers.

6. a) $\frac{1}{6}$ **b)** -4 **c)** 7 **d)** $\frac{3}{5}$ **e)** 7 **f)** $x - y$ **g)** $2x + y$

Self-Test Exercises 5

1)

x	-1	0	1	2
$h(x) = \left(\frac{1}{5}\right)^x$	5	1	$\frac{1}{5}$	$\frac{1}{25}$

x	$\frac{1}{25}$	$\frac{1}{5}$	1	5
$k(x) = \log_{\left(\frac{1}{5}\right)} x$	2	1	0	-1
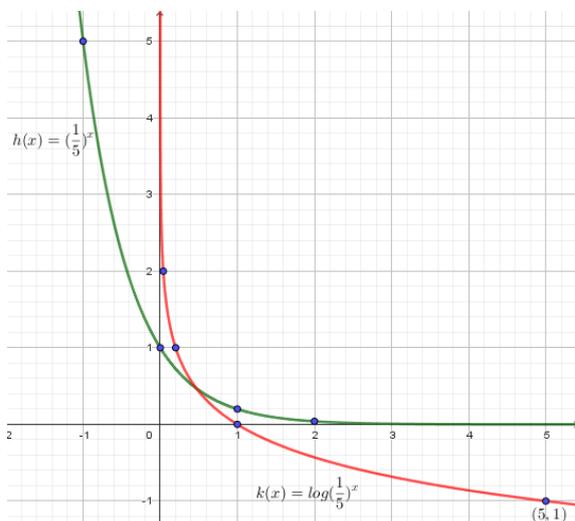
2) The domain of h is the set of all real numbers; the range of h is the set of all positive real numbers

3) The domain of k is the set of all positive real numbers; the range of k is the set of all real numbers

4) The domain of h is the set of all real numbers which is the range of k.

5) The range of h is the set of all positive real numbers which is the domain of k.

Self-Test Exercises 6

1.   **a.** $10^x = 10000$, $10^x = 10^4$ $x = 4$.

    b. $4^{2-3x} = \frac{1}{16}$, $4^{2-3x} = 4^{-2}$, $2 - 3x = -2$. Thus, $x = \frac{4}{3}$.

    c. $3^{3x-1} = 9^{2-x}$, $3^{3x-1} = 3^{2(2-x)}$, $3x - 1 = 4 - 2x$. Thus, $x = 1$.

    d. $\frac{1}{25} = \left(\frac{1}{125}\right)^x$, $\left(\frac{1}{5}\right)^2 = \left(\frac{1}{5}\right)^{3x}$, $2 = 3x$. Thus, $x = \frac{2}{3}$.

    e. $\left(\frac{1}{6}\right)^{3x+2} . 216^{3x} = \frac{1}{216}$, $6^{-3x-2} . 6^{3(3x)} = 6^3$, $6^{-3x-2+9x} = 6^3$, $6x - 2 = 3$, $x = \frac{5}{6}$

f. $3^{2x-2} + 3^{2x} = 10$, $3^{2x} \cdot \frac{1}{9} + 3^{2x} = 10$, $3^{2x}\left(\frac{1}{9} + 1\right) = 10$, $3^{2x} = 9$, $x = 1$

g. $2^x + 3^{x-2} = 3^x - 2^{x+1}$, $2^x + 2(2^x) = 3^x - \left(\frac{1}{9}\right)3^x$, $3(2^x) = \frac{8}{9}(3^x)$, $\left(\frac{2}{3}\right)^x = \left(\frac{2}{3}\right)^3$, $x = 3$

h. $5^{x(x-3)} = \frac{1}{25}$, $5^{x^2-3x} = 5^{-2}$, $x^2 - 3x + 2 = 0$, $x = 1$ or $x = 2$

2. a) $\log_3(2 - 3x) = 2$,

$2 - 3x > 0$ implies $x < \frac{2}{3}$. So, the universe or domain is $\left(-\infty, \frac{2}{3}\right)$.

$\log_3(2 - 3x) = 2$ implies $2 - 3x = 9$.

Hence, $x = -\frac{7}{3}$.

b) $\log_{\sqrt{3}} x = 8$

The universe or domain is $(0, \infty)$.

$\log_{\sqrt{3}} x = 8$, $\frac{\log x}{\log \sqrt{3}} = 8$, $\frac{\log x}{\log 3^{\frac{1}{2}}} = 8$, $\frac{\log x}{\frac{1}{2}\log 3} = 8$, $\frac{\log x}{\log 3} = 4$, $\log_3 x = 4$

So, $x = 81$.

c) $\log_2(2x^2 - x) = 0$

$2x^2 - x > 0$ if and only if $x < 0$ or $x > \frac{1}{2}$.

So, the universe is $(-\infty, 0) \cup \left(\frac{1}{2}, \infty\right)$.

$\log_2(2x^2 - x) = 0$,

$2x^2 - x = 1$,

$2x^2 - x - 1 = 0$,

Therefore, $x = -\frac{1}{2}$, 1. Both are members of the universe.

d) $\log_4 x + \log_4(x - 3) = 1$

$x > 0$ and $x - 3 > 0$ if and only if $x > 3$.

So, the universe is $(3, \infty)$.

$\log_4 x + \log_4(x - 3) = 1$,

$\log_4 x(x - 3) = 1$,

$x^2 - 3x - 4 = 0$,

$(x - 4)(x + 1) = 0$, $x = 4$ or $x = -1$,

$x = 4 \in (3, \infty)$.

e) $\log(4x^2 - 9) - \log(2x + 3) = 1$

$\log \frac{4x^2 - 9}{2x + 3} = \log(2x + 3) = 1$, universe $x > -\frac{3}{2}$ and $x \neq \frac{3}{2}$

$2x + 3 = 10^1 = 10$, $x = \frac{7}{2}$.

f) $\log_6(x + 4) + \log_6(x - 2) = \log_6 4x$

$x + 4 > 0$, $x - 2 > 0$ and $x > 0$

So, the universe is $(2, \infty)$.

$\log_6 \frac{(x+4)(x-2)}{4x} = 0$, $\frac{(x+4)(x-2)}{4x} = 1$, $x^2 + 2x - 8 = 4x$, $x^2 - 2x - 8 = 0$

$x = 4$ or $x = -2$,

But, $x = -2$ is not in the universe. So, $x = 4$.

g) $\log(x + 6) - \log x = \log 2$

$x > 0$ and $x + 6 > 0$ implies $x > 0$.

So, the universe is $(0, \infty)$.

$\log \frac{x+6}{x} = \log 2$, So, $x = 2$.

Self-Test Exercises 7

1. $p = 5{,}000$, $r = 5\% = 0.05$, $n = 4$ and $t = 4$

$A(t) = P \left(1 + \frac{r}{n}\right)^{nt}$

$A(5) = 5{,}000 \left(1 + \frac{5}{400}\right)^{4(5)} = 6099.45$.

Therefore, the ammount in the account will be birr 6,099.45 after 5 years.

2. $20000 = P \left(1 + \frac{0.07}{12}\right)^{12 \times 5} = P(1.0058)^{60} = P(1.4148)$

$P = \frac{20000}{1.4148} = 14136.27$

3. A Population of cockroaches grows exponentially. There are 5 initially and 60 after 2 days. How many are there after 10 days?

$$P(t) = P_0 e^{kt} \text{ where } p(2) = 60 \text{ and } P_0 = 5$$

$$60 = 5e^{2k}$$

$$12 = e^{2k}$$

Taking ln of both sides, $2k = \ln 12$ and $k = \frac{1}{2}\ln 12 = 1.245$ is the growth rate.

$P(10) = 5e^{1.245(10)} = 3{,}469{,}200$ cockroaches after 10 days.

4. $pH = -\log[H^+]$

$4.5 = -\log[H^+]$

$\log[H^+] = -4.5$

$[H^+] = antilog(-4.5) = antilog(5 - 4.5 - 5) = antilog(0.5 - 5)$

$[H^+] = 3.162 \times 10^{-5}$

5. Suppose the hydrogin concentration reading of milk is $3.16 \times 10^{-7} M$. Calculate the pH of the substance and determine whether it is acidic or basic.

$$pH = -\log[H^+]$$

$$pH = -\log 3.16 \times 10^{-7} = -(log 3.16 + (-7)) = -(0.5 - 7) = 6.5 < 7$$

It is acidic.

Answer key to Review Exercises

Unit 1

Part-I True or false

1. True   3. True   5. False

2. False   4. True

Part-II Multiple choice

1. B   3. A   5. B

2. D   4. C

Part-III Fill in the blank space

1. An onto

2. $-3x$

3. **a.** No, because y does not less than y.   **b.** No   **c.** Yes

4. $[4, \infty)$

5. Domain of F= $\{-2, 0, 4, 5\}$ and range of F$\{1, 0, -2, -3\}$.

Part-IV Work out

1. **a.** The domain and the range are the set of all real numbers.

   b. Domain of R is the set of real numbers $x = -3, -1, 0, 1, 3, 4$ and the range of R is the set of real numbers $y = 5, 3, 2, 1, 0, -2$.

   c. Domain of R is the set of real numbers $x: -2 \leq x \leq 2$, and the range of R is the set of real numbers $y: 0 \leq y \leq 2$.

   d. The domain is the set of brothers and sisters in a family and range is the set of sisters in a family.

2.

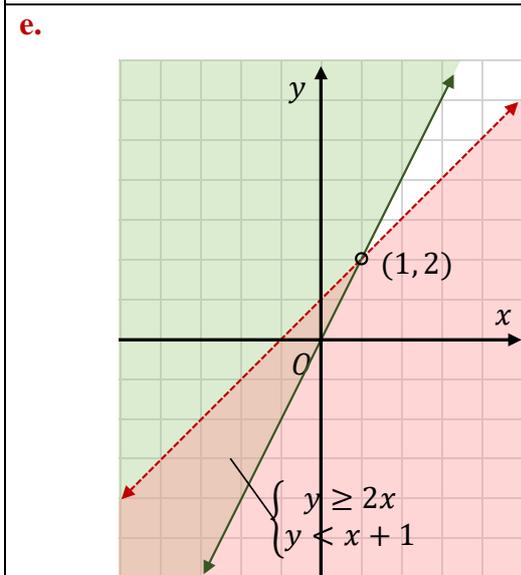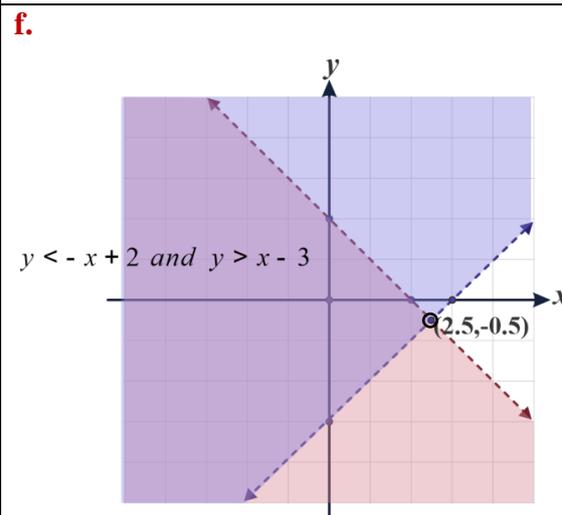| **a.** $y = x^3 - 1$ The domain and range are the set of all real numbers.	**b.** $y \leq x - 2$ The domain and range are the set of all real numbers.
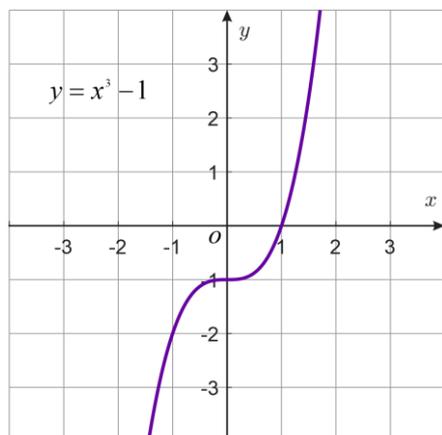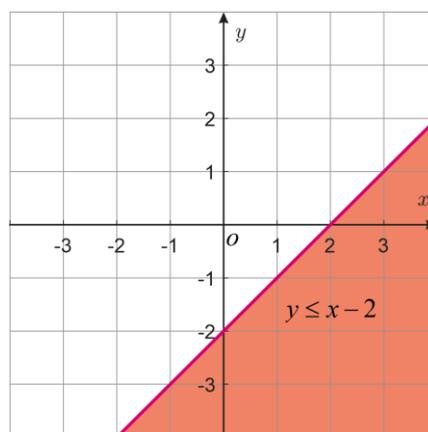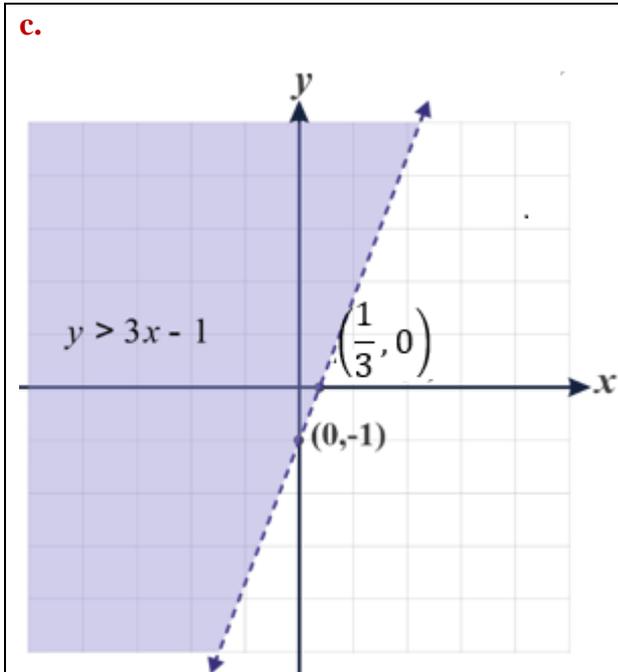
c.

$y > 3x - 1$

$\left(\dfrac{1}{3}, 0\right)$

$(0,-1)$

The domain and range are the set of real numbers.

d.

$y \geq 2x - 3$

$(1.5,0)$

$(0,-3)$

The domain and range are the set of real numbers.

e.

$(1,2)$

$\begin{cases} y \geq 2x \\ y < x + 1 \end{cases}$
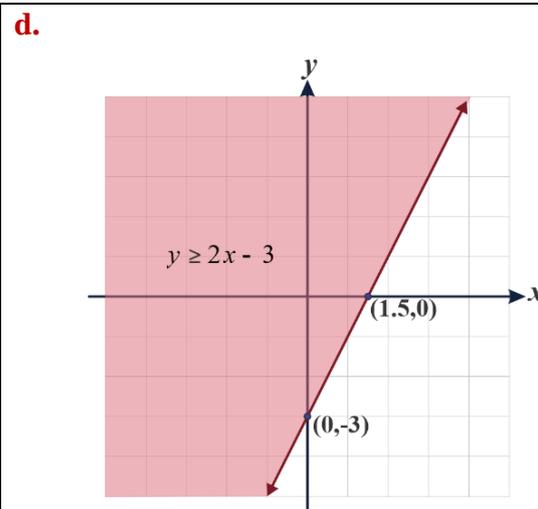
The domain is the set of real numbers $\{x : x < 1, x \in \mathbb{R}\}$

The range is the set of all real numbers $\{y : y < 2, y \in \mathbb{R}\}$.

f.

$y < -x + 2 \; and \; y > x - 3$

$(2.5,-0.5)$

The domain is the set of real numbers $\{x : x < \dfrac{5}{2}, \; x \in \mathbb{R}\}$.

The range is the set of all real numbers.

3. $R = \{(x, y) : y > x - 3 \text{ and } y \leq -x - 1\}$.

The domain is the set of real numbers $x : x > -1$. The range is the set of all real numbers.

4. a. i. $(f + g)(x) = f(x) + g(x) = (2x^2 + 4x - 2) + (-x^2 - x + 7)$

$= 2x^2 - x^2 + 4x - x - 2 + 7.$ Removing bracket.

$= x^2 + 3x + 3.$

ii. $(f - g)(x) = f(x) - g(x) = (2x^2 + 4x - 2) - (-x^2 - x + 7)$

$$= 2x^2 + x^2 - 5x + x - 2 - 7$$

$$= 3x^2 - 4x - 9.$$

iii. $(f + g)(-1) = (-1)^2 + 3(-1) + 3 = 1.$

iv. $(2f - g)(1) = 2(2(1)^2 + 4(1) - 2) - (-(1)^2 - 1 + 7) = 1.$

b. The domain of $f - g$ is the set of all real numbers.

5. Given: $f(x) = 2x^2 + 1$ and $g(x) = x - 3$

  a. **i.** $(fg)(x) = f(x)g(x) = (2x^2 + 1)(x - 3) = 2x^3 - 6x^2 + x + 3.$

    ii. $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{2x^2+1}{x-3}$

    iii. $(fg)(1) = 2(1)^3 - 6(1)^2 + 1 + 3 = 0$

    iv. $\left(\dfrac{f}{g}\right)(2) = \dfrac{2(2)^2+1}{5-3} = \dfrac{9}{2}.$

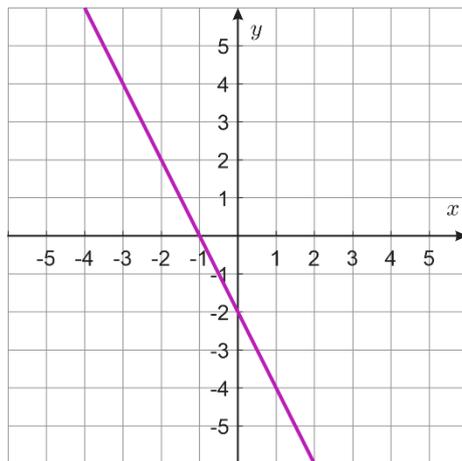 **b.** The domain of $\dfrac{f}{g}$ is the set of all real numbers $\{x: x \neq 3, x \in \mathbb{R}\}$.

6. **a.**

x	-2	-1	0	1	2	3	4
$f(x) = -3x + 4$	10	7	4	1	-2	-5	-8
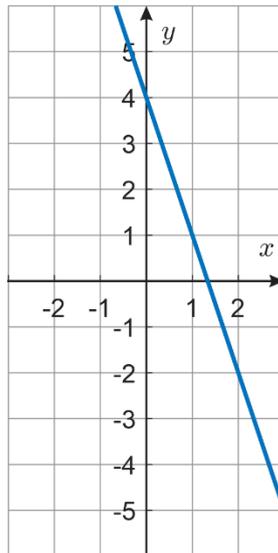
  b.

x	-3	-2	-1	0	1	2	3
$2x + y - 3 = -5$	4	2	0	-2	-4	-6	-8

c.

x	-8	-7	-6	-5	-4	-3	-1	0	1	2	3
$f(x) = 7 - 6x - x^2$	-11	0	7	12	15	16	12	7	0	-9	-20
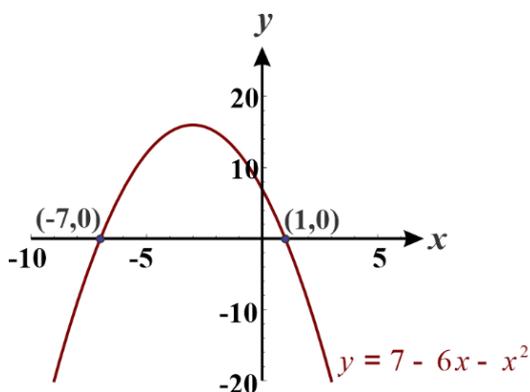
The graph of $y = 7 - 6x - x^2$

7. Consider $C(t) = 3t + 200$. If $t = 4$, then $C(3) = 3 \times 4 + 200 = 212$ Birr is the cost that you pay for the repair.

Unit 2

I. Answer for Multiple Choice problems

1) A **2)** B **3)** A **4)** D **5)** B **6)** A **7)** D

II. Answer for Short answer problems

1. a. quotient $= 2x^2 - 6x + 12$, remainder $= -22$

b. quotient $= 5x + 6$, remainder $= -9x - 21$

c. quotient $= 2x^2 + 4$, remainder $= x^2 + 8x - 9$

d. quotient $= x^3 + x^2 - 3x - 7$, remainder $= 10$

2. a. $k = -\frac{7}{2}$ **b.** $k = -12$

3. $p = -8$ and $q = 4$

4. $f(x) = 2x(x + 1)(x + 2) = 2x^3 + 6x^2 + 4x$

5. a. $-\frac{1}{2}, -\frac{1}{3}$ and 2 **b.** $\frac{1}{3}$ and $\frac{3}{2}$

III. Answer for Workout problems

1. $x - 1$ is a factor implies

$p(1) = a - 6 + b + 6 = 0$

$\quad a + b = 0$

$\quad a = -b$ . . . (*)

$\quad$ And, $p(2) = 8a - 24 + 2b + 6 = -6$

$\quad 8a + 2b = 12$ . . . (**)

$\quad$ Using (*) and (**) $a = 2$ and $b = -2$

2. a. If $1, \ 2$ and -3 are the zeros of $f(x)$, then $x - 1, x - 2$ and $x + 3$ are factors of $f(x)$.

$\quad$ Therefore, $f(x) = k(x - 1)(x - 2)(x + 3)$ for some constant $k \neq 0$.

$\quad$ To find the value of k, we use the given condition $f(-2) = 4$, that is,

$\quad k(x - 1)(x - 2)(x + 3) = 12k = 4$. This implies $k = \frac{1}{3}$.

$\quad$ Hence, the required polynomial function is,

$\quad f(x) = \frac{1}{3}(x - 1)(x - 2)(x + 3) = \frac{1}{3}x^3 - \frac{7}{3}x + 2.$

b. If $0, -\frac{1}{3}, \ \frac{3}{2}$, and 3 are the zeros of $f(x)$, then $x, x + \frac{1}{3}, x - \frac{3}{2}$ and $x - 3$ are factors of $f(x)$.

$\quad$ Therefore, $f(x) = kx(x + \frac{1}{3})(x - \frac{3}{2})(x - 3)$ for some constant $k \neq 0$.

$\quad$ To find the value of k, we use the given condition $f(1) = \frac{3}{4}$, that is,

$\quad kx(x + \frac{1}{3})(x - \frac{3}{2})(x - 3) \ = \frac{4}{3}k = \frac{3}{4}$. This implies $k = \frac{9}{16}$.

$\quad$ Hence, the required polynomial function is,

$\quad f(x) = \frac{9}{16}x\left(x + \frac{1}{3}\right)\left(x - \frac{3}{2}\right)(x - 3) = \frac{9}{16}(x^4 - \frac{25}{6}x^3 + 3x^2 + \frac{3}{2}x).$

3. $8x^6 - 2x^4 + 4x^3 - 30x^2 + 8x + 12$

a. The leading coefficient is $a_6 = 8$ and the constant term is $a_0 = 12$.

$\quad$ Possible values of p are factors of 12. These are $\pm 1, \ \pm 2, \ \pm 3, \pm 4, \pm 6$ and ± 12.

$\quad$ Possible values of q are factors of 8. These are $\pm 1, \ \pm 2, \ \pm 4$ and ± 8.
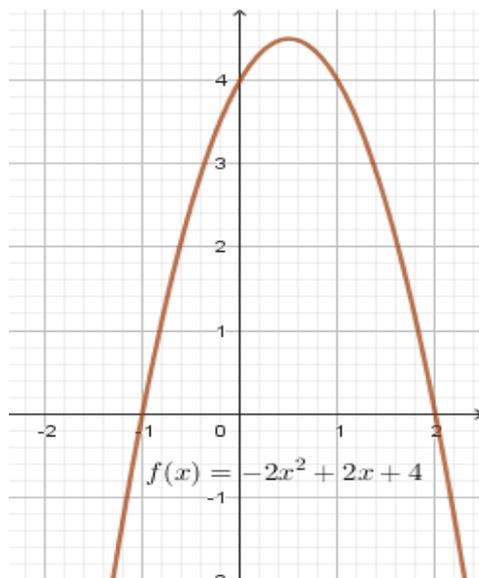
$\quad$ The possible rational zeros $\frac{p}{q}$ are $\pm 1, \ \pm \frac{1}{2}, \ \pm \frac{1}{4}, \ \pm \frac{1}{8}, \ \pm 2, \ \pm 3, \pm \frac{3}{2}, \pm \frac{3}{4}, \pm \frac{3}{8} \pm 4, \pm 6 \pm$

$\quad 12, \pm 6$. That is, $f(x)$ has 26 possible rational zeros. To find the actual rational zeros you can check by substituting each in $f(x)$ for x.
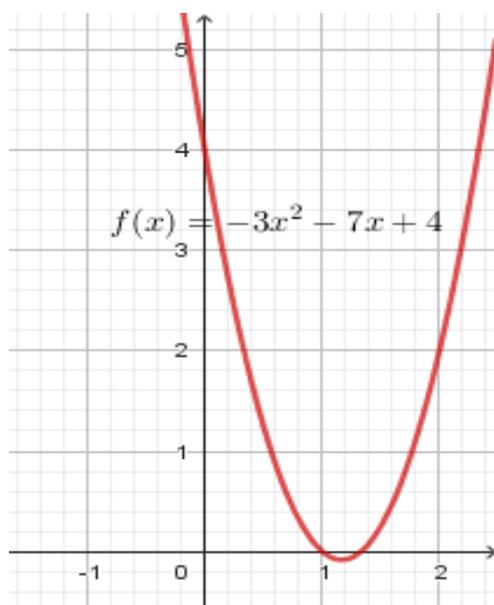
$\quad f\left(-\frac{1}{2}\right) = f(2) = f\left(-\frac{3}{4}\right) = f\left(\frac{2}{3}\right) = 0.$

$\quad$ Therefore, $-\frac{1}{2}, 2, -\frac{3}{4}$, and $\frac{2}{3}$ are the only rational zeros.

4. a

$f(x) = -2x^2 + 2x + 4$

b.

$f(x) = -3x^2 - 7x + 4$

5. Let the two numbers be x and y. Then,

$x - y = 22$ . . . (1)

If we denote the product of the two numbers by A, then

$A = xy$ . . . (2)

To find the minimum value of A, first we solve either for x or for y from equation (1).

Solving for y from equation (1) gives $y = x - 22$ and substituting this in equation (2) gives

$A = x(x - 22) = x^2 - 22x,$

$= (x^2 - 22x + 121) - 121,$

$= (x - 11)^2 - 121.$

$= -121 + (x - 11)^2$

Since, $(x - 11)^2 \geq 0$

$A = -121 + (x - 11)^2 \geq -121.$

Therefore, -121 is the minimum product and this minimum product is obtained when the value of $x = 11$. And when $x = 11$, $y = x - 22 = -11$. That is, $y = -11$.

6. Find the two real numbers whose sum is 16 and whose product is the maximum.

Let the two numbers be x and y. Then,

$x + y = 16$ . . . (1)

If we denote the product of the two numbers by A, then

$A = xy$ . . . (2)

To find the maximum value of A, first we solve either for x or for y from equation (1).

Solving for y from equation (1) gives $y = 16 - x$ and substituting this in equation (2) gives

$A = x(16 - x) = -x^2 + 16x = -(x^2 - 16x),$

$\qquad = -(x^2 - 16x + 64) + 64,$

$\qquad = -(x - 8)^2 + 64.$

$\qquad = 64 - (x - 8)^2$

Since, $(x - 8)^2 \geq 0$

$A = 64 - (x - 8)^2 \leq 64.$

Therefore, 64 is the maximum product and this maximum product is obtained when the value of $x = 8$. And when $x = 8$, $y = 16 - x = 8$. That is, $y = 8$.

Unit 3

I. Answer for Multiple Choice problems

1) D   **2.** B   **3.** A   **4.** D   **5.** A   **6.** A   **7.** D

II. Answer for Short answer Problems

1) $6^2 = 36$ **2)** $\log_5\left(\frac{1}{125}\right) = -3$ **3)** $p + 3q$ **4)** 16

5) a. $x = -\frac{11}{5}$ **b.** 0   **c.** $x = -1$ **d.** $x = \log_2 3$

6)   **a)** $3 - 2x > 0$ implies $x < \frac{3}{2}$. So, the universe is $\{x: x < \frac{3}{2}\}$

and the solution is $x = -\frac{13}{2}$.

b) The universe is $(5, \infty)$ and the solution is $x = 6$.

c) The universe is $x > 4$ and the solution is $x = \frac{33}{7}$.

d) The universe is $(0, \infty)$ and the solution is $x = \sqrt{10}$ or $x = \frac{1}{\sqrt[3]{10}}$.

III. Answer for Workout Problems

1.   **a)** $4^{x+1} = 8 - 2^{x+2}$

$\qquad (2^2)^{x+1} + 2^{x+2} - 8 = 0$

$\qquad 4 \cdot (2^x)^2 + 4 \cdot 2^x - 8 = 0$

$$(2^x)^2 + 2^x - 2 = 0$$

Here, let $2^x = y \ (y > 0)$

Then, $y^2 + y - 2 = 0, (y + 2)(y - 1) = 0, y = -2, 1$

Since $y > 0, y = 1$

Hence, $2^x = 1, \ x = 0$

b) $\dfrac{3^x}{27^{2x-1}} = 9^{x+6}$

$$\dfrac{3^x}{3^{3(2x-1)}} = 3^{2(x+6)}$$

$$x - 3(2x - 1) = 2(x + 6)$$

$$x = -\dfrac{9}{7}$$

c) $2^{4x+1} = \log_2 64 + 2^{2(x+1)}$

$2.(2^{2x})^2 = 6 + 4.2^{2x}$, let $y = 2^{2x}, y > 0$

implies $2y^2 - 4y - 6 = 0, \ (2y + 2)(y - 3) = 0, y = -1$ or $y = 3$

$2^{2x} = 3$ implies $x = \log_2 \sqrt{3}$

d) $\dfrac{16^{(5-2x)} \times 32^{(x-2)}}{4^{(5x-1)}} = \dfrac{1}{2}$

$$2^{4(5-2x)+5(x-2)-2(5x-1)} = 2^{-1}$$

$$4(5 - 2x) + 5(x - 2) - 2(5x - 1) = -1$$

$$-13x = -13, \ x = 1$$

2. a) The universe is $(0, \infty)$.

$$\log_3(2x + 5) = 1 - \log_3 x$$

$$\log_3(2x + 5) + \log_3 x = 1$$

$$\log_3 x(2x + 5) = 1$$

$$2x^2 + 5x - 3 = 0$$

$$x = \dfrac{1}{2}$$

b) The universe is $(1, \infty)$.

$$2 \log(2x - 2) = \log(4x^2 + 3)$$

$$\log(2x - 2)^2 = \log(4x^2 + 3)$$

$$(2x - 2)^2 = 4x^2 + 3$$

$$4x^2 - 8x + 4 = 4x^2 + 3$$

$$x = -\dfrac{1}{8}$$

c) The universe is $\left(\dfrac{1}{8}, \infty\right)$.

$$\log_2(8x - 1) - 2\log_2(x + 1) = 3 - \log_2(x + 4)$$

$$\log_2\left(\frac{8x-1}{(x+1)^2}\right)(x+4) = 3$$

$$x = \frac{12}{15}$$

d) The universe is $(0, \infty)$.

$$2\log_2\left(\frac{x}{2}\right) + \log_2\sqrt{x} = 8$$

$$2(\log_2 x - \log_2 2) + \frac{1}{2}\log_2 x = 8$$

Let $\log_2 x = y$, then $2y - 2 + \frac{1}{2}y = 8$

$y = 4$ implies $\log_2 x = 4$

$x = 16$

e) The universe is $(1, \infty)$ and the solutions are $x = 10$ and $x = 10,000$.

$$2\log_2 x + \log_2(x - 1) - \log_2(5x + 4) = 1$$

$$\log_2\left(\frac{x^2(x-1)}{5x+4}\right) = 1$$

$$\frac{x^2(x-1)}{5x+4} = 2$$

$$x^3 - x^2 = 10x + 8$$

$$x^3 - x^2 - 10x - 8 = 0$$

$$(x+1)(x-4)(x+2) = 0$$

$$x = 4$$

f) The universe is $(0, 6)\backslash\{1\}$ and the solution is $x = 4$.

$$\frac{1}{\log_x\sqrt{3}} = \log_3(6 - x)$$

$$\log_3(6 - x) \cdot \log_x\sqrt{3} = 1$$

$$\log_3(6 - x) \cdot \frac{\log_3\sqrt{3}}{\log_3 x} = 1$$

$$\log_3(6 - x) \cdot \left(\frac{1}{2}\right) = \log_3 x$$

$$\frac{1}{2}\log_3(6 - x) = \log_3 x$$

$$\log_3(6 - x)^{\frac{1}{2}} = \log_3 x$$

Then,

$$(6 - x)^{\frac{1}{2}} = x$$
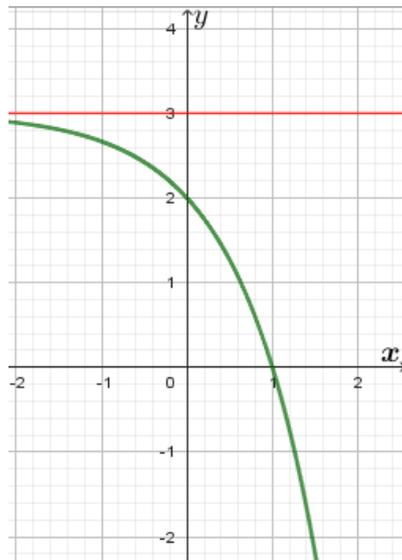
Squaring both side of the equation,
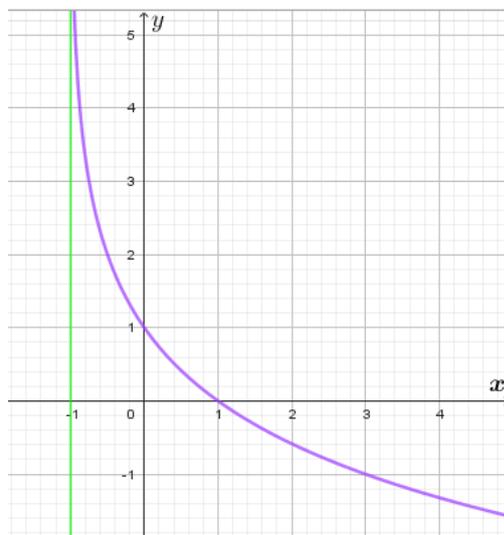
$$x^2 + x - 6 = 0$$

$$(x+3)(x-2) = 0, x = -3, x = 2$$

$$x = 2$$

3.   **a)** $\log_a 20 = \log_a 4 \times 5 = \log_a 4 + \log_a 5 = 2m + n$

b) $\log_a 0.4 = \log_a \frac{4}{10} = 2\log_a 2 - \log_a(2 \times 5) = m - n$

c) $\log_a 3.2 = \log_a \frac{16}{5} = 4\log_a 2 - \log_a 5 = 4m - n$

d) $\log_4 5 = \frac{\log_a 5}{\log_a 4} = \frac{n}{2m}$

4. a) has x -intercept $=(1,0)$, y- intercept $= (0,2)$, the asympyote is the line $y = 3$, the domain is all real numbers and the range is $\{y : y < 3\}$.

b) x-intercept $= (1, 0)$, y- intercept $(1, 0)$, the asympyote is the line $x = -1$, the domain is $\{x : x > -1\}$ and the range is all real numbers.

5. a) $P(t) = P_0 e^{kt}$, where $P_0 = 10{,}000, p(1) = 30{,}000$

$P(1) = 10{,}000 e^{k(1)} = 30{,}000,$

$e^k = 3,$

$\ln e^k = \ln 3,$

$k = \ln 3 \approx 1.0986,$

$P(t) = 10{,}000e^{1.0986t}$ is the function that models the population.

The doubling period is when the number of bacteria is 20,000

$$10{,}000e^{1.0986t} = 20{,}000$$

$$e^{1.0986t} = 2$$

$$\ln e^{1.0986t} = \ln 2$$

$$1.0986t = \ln 2$$

$$t = \frac{\ln 2}{1.0986}$$

$$t = 0.6309 \text{ hour}$$

b) $t = 3.$

$P(3) = 10{,}000e^{1.0986(3)} = 10{,}000(27) \approx 270{,}000$

Therefore, the population will be $270{,}000$ after 3 hrs.

6. Here $P = 20{,}000$, $r = 5.25\% = 0.0525$, $n = 2$, $t = 10$

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} = 20{,}000\left(1 + \frac{0.025}{2}\right)^{2 \times 10} = 20{,}000(1.0125)^{20} = 25640.74$$

Here $P = 10{,}000$, $r = 7\% = 0.07$, $n = 365$, $t = 5$.

$$A(5) = P\left(1 + \frac{r}{n}\right)^{nt} = 10{,}000\left(1 + \frac{0.07}{365}\right)^{365(5)}$$

$$= 10{,}000(1.001918)^{1825},$$

7. $A(t) = P\left(1 + \frac{r}{n}\right)^{nt},$

$A(4) = 3{,}000\left(1 + \frac{0.0775}{12}\right)^{12 \times 4} = 4{,}086.2,$

$4{,}086.2 - 3000 = 1086.2$

So, the amount of interest earned at the end of four years is birr $1{,}086.2$.